

Circulation Sensitivity to Heating in a Simple Model of Baroclinic Turbulence

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ABSTRACT

This paper examines the sensitivity of the circulation of an idealized primitive equation two-level model on the form and strength of the heating, aiming to understand the qualitatively different sensitivity of the isentropic slope on differential heating reported by previous idealized studies when different model formulations are used. It is argued that this contrasting behavior might arise from differences in the internal determination of the heating. To test this contention, the two-level model is forced using two different heating formulations: a standard Newtonian cooling formulation and a highly simplified formulation in which the net lower-to-upper troposphere heat transport is prescribed by construction. The results are interpreted using quasigeostrophic turbulent closures, which have previously been shown to have predictive power for the model. It is found that the strength of the circulation, as measured by eddy length and velocity scales and by the strength of the energy cycle, scales with the vertical heating (the lower-to-upper troposphere heat transport), with a weak dependence. By contrast, the isentropic slope is only sensitive to the structure of the heating, as measured by the ratio between meridional versus vertical heating, and not to the actual strength of the heating. In general the heating is internally determined, and this ratio may either increase or decrease as the circulation strengthens. It is shown that the sign of the sensitivity depends on the steepness of the relation between vertical heating and stratification for the particular heating formulation used. The quasigeostrophic limit (fixed stratification) and the prescribed heating model constrain the possible range of behaviors and provide bounds of sensitivity for the model. These results may help explain the different sensitivity of the isentropic slope on differential heating for dry and moist models and for quasigeostrophic and primitive equation models.

1. Introduction

The determination of the mean extratropical thermal structure is a longstanding problem in the general circulation of the atmosphere. The equilibrium extratropical climate arises from the competition between diabatic heating and dynamical transport, both players being in general a function of the time-dependent state vector. Yet while the heating is dominated by its linear part (one can get a good approximation to the mean heating using the mean temperature alone, at least in a dry model), the bulk of the dynamical forcing is nonlinear and results from the correlation between departures from the time

mean. The closure problem requires relating these eddy fluxes to the mean state.

This is of course a very hard problem. In fact, it could even be ill posed, as different eddy fluxes could in principle be consistent with the same mean state. Indeed, one possible closure is baroclinic adjustment (Stone 1978), which assumes that the system has some preferred equilibria toward which it evolves regardless of the heating (see Zurita-Gotor and Lindzen 2007 for a review). In that case, the eddy fluxes must vary with the heating and conspire to keep an invariable mean state, so that the fluxes are a function of the heating rather than a function of the mean state. Some studies have suggested that this paradigm might be relevant in the atmosphere (Stone 1978) and in idealized models, both primitive equation (PE; Schneider 2004) and quasigeostrophic (QG; Stone and Branscome 1992; Welch and Tung 1998). However, other studies do not seem to conform to the baroclinic

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adjustment paradigm [e.g., Salmon (1980) and Vallis (1988) in the QG case and Zurita-Gotor (2008) in primitive equations].

An alternative approach to the closure problem is turbulent diffusion, which predicts smooth dependence of the fluxes on the mean state. Using mixing length assumptions, the diffusivity is essentially a function of the scale of the eddies, and the classical phenomenology of quasigeostrophic turbulence predicts that this could be larger than the baroclinic instability scale because of the existence of an inverse cascade (Salmon 1980). Assuming that the inverse cascade is halted by the beta effect, Held and Larichev (1996) obtain explicit analytical predictions for the eddy scales as a function of the mean flow parameters for the two-level quasigeostrophic model. This leads to the following closure for the eddy meridional heat flux:

$$\overline{v'\theta'} \approx -D\partial_y \bar{\theta} \sim -\beta\lambda^3 \xi^3 \partial_y \bar{\theta}, \quad (1)$$

where D is diffusivity, $\lambda = NH/f$ is the Rossby radius, $\xi = -(f/\beta H)\partial_y \bar{\theta}/\partial_z \bar{\theta}$ is the criticality (a measure of the isentropic slope), N is the buoyancy frequency, H is the fluid depth, and f is the Coriolis parameter and β its meridional derivative. Zurita-Gotor (2007) shows that this closure works reasonably well for the two-layer quasigeostrophic model, and notes that the steep sensitivity of the fluxes on ξ predicted by the closure implies a very weak sensitivity of the mean state on the heating. This makes it hard to distinguish between the diffusive closure and the baroclinic adjustment paradigm of a preferred constant ξ . Zurita-Gotor and Vallis (2009, hereafter ZV09) test the quasigeostrophic closure in a two-level primitive equation model, in which the stratification is internally determined. They show that the closure also works well in that case when the stratification is diagnosed from the model.

In this study we address the other facet of the equilibration problem. Regarding the diffusive closure as truth, based on the above studies, we investigate the implications of this closure for the sensitivity of the mean state on the heating. Our motivation is to understand the seemingly different behavior of the criticality parameter in dry primitive equation models when different heating formulations are used. While in models forced by Newtonian cooling the criticality–isentropic slope remains fairly robust and of $O(1)$ as the forcing is varied (Schneider 2004), in models forced by gray radiation the extratropical stratification is nearly neutral over much of the troposphere (Frierson et al. 2006), implying large criticalities in the dry limit. It seems plausible that this difference could be due to the stronger degree of convective destabilization in the gray radiation model, forced from below,

compared to the Newtonian cooling model for typical parameters. Yet, to add confusion, the gray radiation model of Schneider and O’Gorman (2008) seems to behave differently from that of Frierson et al. (2006) in that it also has a well-defined stratification and order-one criticalities in the (cold) dry limit. The reasons for these differences are not clear but it seems plausible that they could be due to differences in the heating formulation, for instance the inclusion of atmospheric shortwave absorption in the model of Schneider and O’Gorman (2008) and the small optical depth used in the dry limit of the same model. The inclusion of moisture affects the isentropic slope in similar ways in both models. While consideration of moisture introduces complexity at many levels, it is worth asking whether its effects can be understood, at least qualitatively, simply by taking into account how moisture impacts the heating.

Another feature that we would like to clarify with this study is the strikingly different sensitivity of the criticality on differential heating for the quasigeostrophic and primitive equation models when both are forced using Newtonian cooling. In the quasigeostrophic case, the criticality always increases with differential heating. This is in fact a trivial result from Eq. (1), which predicts

$$Q_H \sim \frac{\partial_z \bar{\theta} H \beta^2 \lambda^3}{f} \xi^4, \quad (2)$$

where Q_H is the differential heating,¹ a measure of the global energy transport from low to high latitudes, which scales as the meridional eddy heat flux. Since λ , H , and $\partial_z \bar{\theta}$ are all fixed in quasigeostrophic theory, the criticality must increase with differential heating in this model, albeit with a weak dependence $\xi \propto Q_H^{1/4}$. As a result, an often-used procedure to increase the criticality of two-layer quasigeostrophic flow is to increase the baroclinicity of the radiative-equilibrium profile, which effectively increases the differential heating Q_H .

This is in contrast with the two-level primitive equation model, in which the criticality typically decreases when increasing the baroclinicity of the radiative-equilibrium profile (ZV09). Held (2007) discuss how this happens for the particular case of a convectively neutral radiative equilibrium. Yet for other values of the radiative-equilibrium stratification (the control set in ZV09), the same model is found to exhibit very little sensitivity on radiative-equilibrium baroclinicity, which is more consistent with the multilevel results of Schneider (2004).

¹ See section 2 for definitions and conventions.

Finally, the criticality may also increase with baroclinicity in the same model when the radiative-equilibrium stratification is sufficiently large, consistent with the quasigeostrophic sensitivity. This seems reasonable because in that limit the stratification should not depart much from its radiative-equilibrium value.

This paper aims to understand all these apparently conflicting results by describing the sensitivity of the two-level primitive equation model of ZV09 on the heating. We force the model using two different heating formulations, very different qualitatively: a standard model forced by Newtonian cooling and a model forced from below, in which the net vertical heat transport is prescribed by construction. The latter is intended to serve as a simplified two-level version of the gray radiation model of Frierson et al. (2006). We compare how both models equilibrate when the external parameters defining the heating are varied. The paper is structured as follows: Section 2 presents the theoretical background and derives explicit expressions for the mean temperature gradients as a function of the heating, based on the diffusive closure of Held and Larichev (1996). Section 3 introduces the prescribed heating model and describes the sensitivity of its mean climate on the heating. Section 4 describes the sensitivity of the Newtonian cooling model. Section 5 closes with some concluding remarks and speculation on how our results could be extrapolated to more realistic models.

2. Diffusive closure and the maintenance of the mean state

We use as a framework for this paper the two-level, primitive equation, beta-plane, hydrostatic, Boussinesq model of ZV09. Details about the model formulation, including the vertical discretization, can be found in that paper. The model is most similar to the standard quasigeostrophic two-layer model, except for the determination of the stratification. ZV09 demonstrate that quasigeostrophic theory is relevant for this model once the stratification is known.

Our starting point is the diffusive closure for the meridional eddy heat flux of Held and Larichev (1996). Although this closure was originally derived for the high-criticality limit, Zurita-Gotor (2007) and ZV09 have shown that it also works well at moderate criticalities when applied locally. This is in contrast with doubly periodic results (Lapeyre and Held 2003), for which the empirical closure steepens at low criticality when using domain-averaged diagnostics. Without this steepening, the local closure is given by Eq. (1).

To evaluate the local diffusive closure, a reference latitude and height are also needed and we use for this

purpose the latitude of maximum vertically integrated eddy meridional heat flux and a midtropospheric level (midtropospheric values and vertical averages are equivalent in the two-level discretization). Note that this convention is slightly different from that used by ZV09, who chose the latitude of maximum lower-level wind as the reference latitude. The new convention is more convenient for the theory developed in this section, but the diffusive closures work a bit better when using the old convention.

In the nonquasigeostrophic case, a closure for the eddy vertical heat flux is also needed. Following Held (2007) and ZV09, one can derive that closure assuming that the eddies are adiabatic. Constructing a balance equation for θ'^2 —

$$\overline{v'\theta'}\partial_y\bar{\theta} + \overline{w'\theta'}\partial_z\bar{\theta} \approx \overline{Q'\theta'}, \quad (3)$$

where the combined advection of $\bar{\theta}'^2$ by eddies and mean is neglected based on the assumption that the generation of eddy potential temperature variance is locally balanced by its destruction (see ZV09 for details)—and assuming that the eddies are adiabatic ($Q' = 0$, where Q is the heating), one finds that the mixing slope should scale with the isentropic slope ($\overline{w'\theta'}/\overline{v'\theta'} \sim -\partial_y\bar{\theta}/\partial_z\bar{\theta}$). This allows us to write a closure for the eddy vertical heat flux as a function of the eddy meridional heat flux:

$$\overline{w'\theta'} \approx -\overline{v'\theta'} \frac{\partial_y\bar{\theta}}{\partial_z\bar{\theta}} = \frac{\Theta_0}{g} \beta^3 \lambda^5 \xi^5, \quad (4)$$

where Eq. (1) was used. Here Θ_0 is a reference temperature and the Rossby radius λ is evaluated using the full model depth H .

ZV09 tested compliance with the closures of [Eqs. (1) and (4)] in a model forced with Newtonian cooling as the external parameters were varied, finding reasonable agreement. In this paper, we will extend that previous work by investigating the implications of the closures for the mean state in the forced-dissipative system.

With this aim, consider the time- and zonal-mean thermodynamic equation in discretized form for our two-level model:

$$\frac{\partial}{\partial y} \overline{v'_k\theta'_k} + \frac{\partial}{\partial y} \bar{v}_k \bar{\theta}_k - (-1)^k \frac{2}{H} (\overline{w'\theta'} + \bar{w}\bar{\theta}) = \overline{Q_k}, \quad (5)$$

where $k = 1$ (2) in the lower (upper) level and w is calculated at midlevel (vertical heat fluxes are calculated interpolating θ to midlevel as well).

Summing this equation over both levels, and then integrating meridionally between the southern boundary

and the latitude y_m of maximum vertically integrated meridional heat flux, we obtain

$$\overline{v'\theta'}\left(y_m, \frac{H}{2}\right) = \frac{1}{2} \int_{-L/2}^{y_m} (\overline{Q}_1 + \overline{Q}_2) dy = \mathcal{Q}_H. \quad (6)$$

We neglected above the meridional advection by the mean meridional circulation (MMC), which is typically smaller than the eddy heat flux. When this term is not negligible, for instance at low criticality, it may be lumped into $\mathcal{Q}_H$.

Since the divergence of the meridional heat transport vanishes at y_m , the vertically integrated heating changes sign at that latitude and the parameter $\mathcal{Q}_H$ may be interpreted as the net low-latitude heating or net high-latitude cooling (both being equal, of course). We will refer to this parameter as *the differential heating*.

Likewise, we can relate $\overline{w'\theta'}(y_m, H/2)$ to the heating by evaluating Eq. (5) at $y = y_m$ and subtracting the result over both levels. This gives

$$\overline{w'\theta'}\left(y_m, \frac{H}{2}\right) \approx \frac{H}{4} [\overline{Q}_1(y_m) - \overline{Q}_2(y_m)] = \mathcal{Q}_V. \quad (7)$$

We neglected above the heating by the mean meridional circulation, both meridional and vertical. If the location of maximum meridional eddy heat flux is close to that of maximum eddy momentum flux convergence, then one expects $f\overline{v}$ to be maximized around y_m and $\overline{w}$ to be small. We also neglected the tilt in the eddy meridional heat flux; that is, we assumed that $\partial_y \overline{v'\theta'}$ vanishes in both levels and not only in a vertical average. When any of these terms is not negligible, it may be lumped into $\mathcal{Q}_V$ as an additional heating contribution. The parameter $\mathcal{Q}_V$ measures the lower-level heating minus upper-level cooling at the latitude y_m or, since the vertically integrated heating vanishes at that location by definition, this is twice the lower-level heating or upper-level cooling. We will refer to $\mathcal{Q}_V$ as *the vertical destabilization*.

Per our conventions, these eddy fluxes $\overline{v'\theta'}(y_m, H/2)$ and $\overline{w'\theta'}(y_m, H/2)$ are the same that were expressed as a function of the mean state in the closure Eqs. (1) and (4). For simplicity, we shall drop henceforth references to the location where these fluxes are evaluated and refer to them simply as $\overline{v'\theta'}$, $\overline{w'\theta'}$.

From Eqs. (1), (4), (6), and (7), we can relate the heating and mean state as follows:

$$\mathcal{Q}_V \sim \overline{w'\theta'} \sim \frac{\Theta_0}{g} \beta^3 \lambda^5 \xi^5, \quad (8a)$$

$$\frac{\mathcal{Q}_V}{\mathcal{Q}_H} \sim \frac{\overline{w'\theta'}}{\overline{v'\theta'}} \sim \frac{\beta H}{f} \xi. \quad (8b)$$

We can also invert these expressions to express the criticality and Rossby radius in terms of the heating:

$$\xi \sim \frac{f}{\beta H} \frac{\mathcal{Q}_V}{\mathcal{Q}_H}, \quad (9a)$$

$$\lambda \sim \left(\frac{g}{\Theta_0}\right)^{1/5} \frac{\beta^{2/5} H}{f} \frac{\mathcal{Q}_H}{\mathcal{Q}_V^{4/5}}. \quad (9b)$$

The following implications are noteworthy (assuming all parameters but $\partial_y \overline{\theta}$ and $\partial_z \overline{\theta}$ are fixed):

- The strength of the energy cycle, proportional to $\overline{w'\theta'}$, depends only on the vertical destabilization $\mathcal{Q}_V$, and not on the differential heating.
- The criticality depends only on the structure of the heating, through the ratio between vertical destabilization and differential heating. This result stems directly from the requirement that the full eddy heat flux be aligned along the isentropic slope.
- In the PE case, when λ can change, it is the full product $\xi\lambda$ and not just ξ that is stiff against changes in the forcing. In particular, $\xi\lambda \sim \mathcal{Q}_V^{1/5}$. Note that $\xi\lambda$ is an estimate of the eddy length scale in the theory of Held and Larichev (1996).

Finally, it is convenient for later reference to express the temperature gradients in terms of the heating:

$$\partial_y \overline{\theta} \sim \left(\frac{\Theta_0}{g}\right)^{3/5} \beta^{4/5} \frac{\mathcal{Q}_H}{\mathcal{Q}_V^{3/5}}, \quad (10a)$$

$$\partial_z \overline{\theta} \sim \left(\frac{\Theta_0}{g}\right)^{3/5} \beta^{4/5} \frac{\mathcal{Q}_H^2}{\mathcal{Q}_V^{8/5}}. \quad (10b)$$

The above analysis implies that knowledge of the heating fully determines the local thermal structure. However, in general the heating is not known a priori but is itself coupled to the dynamics. In the next two sections we will explore the sensitivity of the mean state on the heating for two different heating formulations.

3. A prescribed heating model

On the earth, the bulk of the incoming shortwave radiation is absorbed at the surface, with the heating reaching the interior atmosphere mainly through radiative and convective fluxes emanating from the surface. The vertical transport by the mean and eddy motions and by convection carries this heat upward to higher atmospheric levels, where it is radiated away. For this reason, the atmosphere is said to be “heated from below.” Idealized numerical models also heated from below (Frierson et al. 2006, 2007) tend to be less stably stratified than the more

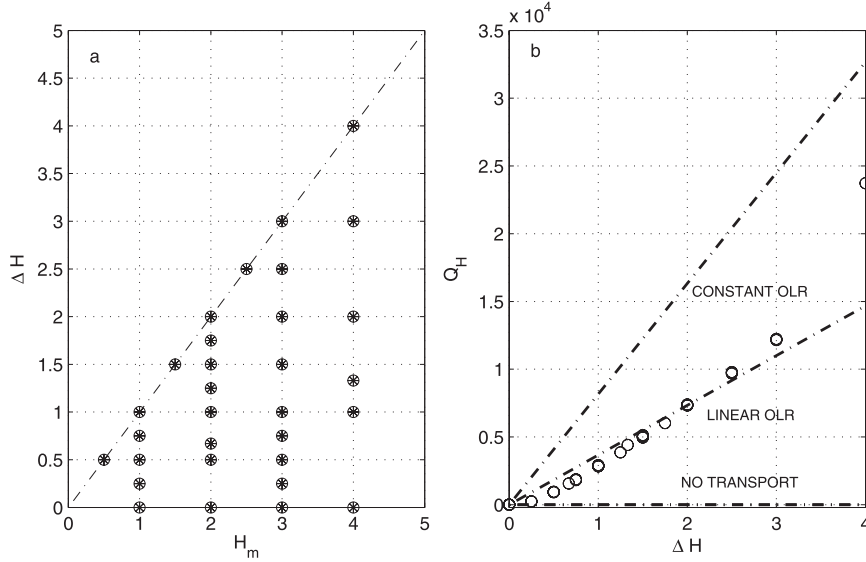


FIG. 1. (a) Combinations of $(H_m, \Delta H)$ for all experiments performed. (b) Sensitivity of the differential heating (diagnosed) on the parameter ΔH for all simulations, and analytical predictions assuming a constant OLR, a linear OLR, or no meridional heat transport.

widely used Newtonian cooling models, at least for typical parameters of the latter.

In this section, we design a heating formulation for our two-level model aimed at replicating this type of heating. In our model, all the heating occurs in the lower level and all the cooling occurs in the upper level. The lower-level heating is prescribed and defined as a function of latitude by the following functional form:

$$H_1 = H_m - \Delta H \times \tanh\left[\frac{y - (L_Y/2)}{\sigma}\right]. \quad (11)$$

The heating has mean value H_m at the center of the channel and changes by $\pm\Delta H$ as one moves from low to high latitudes over a baroclinic zone with characteristic width σ ($L_Y = 18 \times 10^3$ km and $\sigma = 1200$ km are fixed in this paper). We assume that $\Delta H \leq H_m$, so that H_1 remains positive definite. When $\Delta H = H_m$, the heating goes to zero in the northern part of the domain. Figure 1a shows the combinations of $(H_m, \Delta H)$ used in the experiments described below.

In the upper level, the cooling is aimed to represent infrared cooling, which we linearize in the following form:

$$C_2 = -\frac{1}{\tau}(T_2 - T_{\text{rad}}), \quad (12)$$

where we take a constant $T_{\text{rad}} = 273$ K for simplicity and keep $\tau = 20$ days also constant for the simulations described below.

Because all the cooling occurs in the upper level in this model, there must be in equilibrium an upward transport of heat from the lower to the upper level exactly equal—in a global average—to the net lower-level heating. This vertical heat transport may be accomplished by either convection or baroclinic eddies. As we shall see, grid-scale convection is very efficient in rendering the time-mean state statically neutral when baroclinic eddies are weak or absent, even without a convective scheme. Likewise, the mean upper-level temperature must satisfy $\langle T_2 \rangle = T_{\text{rad}} + \tau \langle H_1 \rangle = T_{\text{rad}} + \tau H_m$ so that the cooling rate equals the prescribed heating rate. As the lower-level heating H_m increases, the vertical heat flux by the fluid motion increases by the same amount because there are no infrared radiative feedbacks in this model.²

Since there are no infrared radiative feedbacks in this model, the vertical destabilization is roughly prescribed by construction: $Q_V \approx H_m$. We neglect the small corrections that arise when y_m is shifted from midchannel (in which case the heating at y_m differs from the mean heating H_m) or from contributions to Q_V by the MMC heating or the tilt in $\overline{v'\theta'}$. In contrast, the differential heating Q_H is controlled by the parameter ΔH in this model, but is ultimately internally determined. For the

² This is both unrealistic and different from the true gray radiation model, in which the infrared fluxes increase with temperature and compensate some of the vertical destabilization when the surface heating increases. The elimination of this feedback exaggerates the vertical destabilization in our model and facilitates the interpretation of our results.

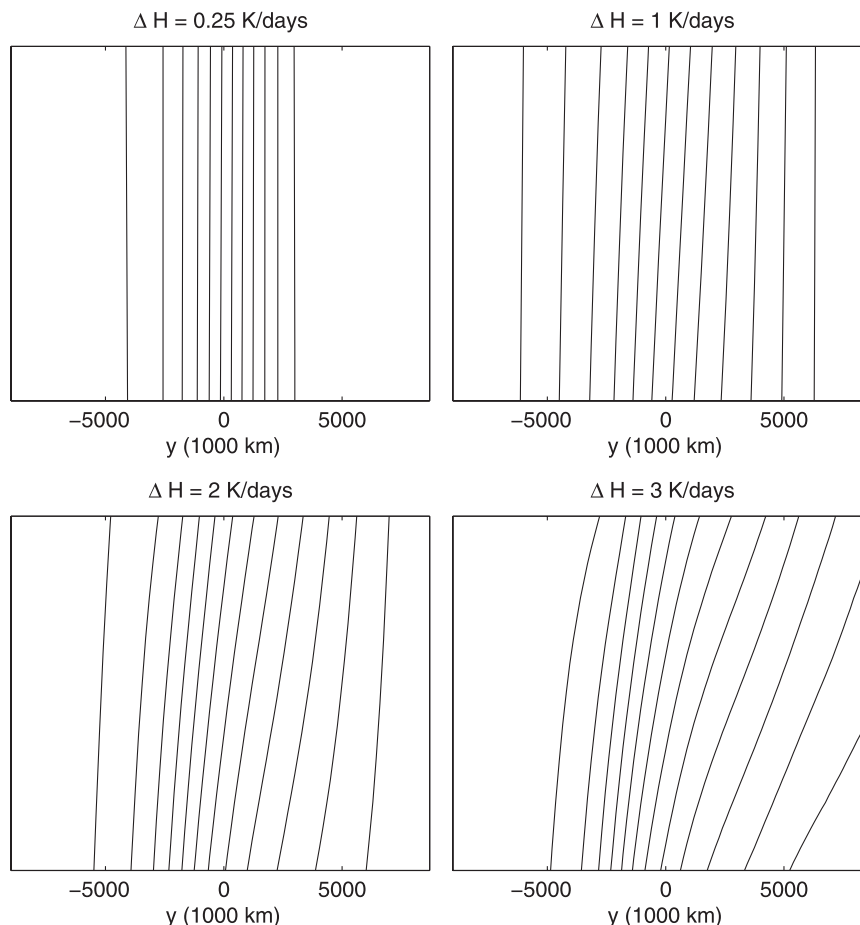


FIG. 2. Contours of time-mean potential temperature for simulations with fixed H_m ($H_m = 3 \text{ K day}^{-1}$) and varying ΔH , as indicated.

same ΔH , one can have $\mathcal{Q}_H = 0$ if the lower-level heating is “radiated away” locally without any meridional transport [corresponding to maximum temperature gradients in the upper level: $T_2(y) = T_{\text{rad}} + \tau H_1(y)$] or a maximum $\mathcal{Q}_H$ when the upper-level temperature is flat [a constant outgoing longwave radiation (OLR) limit]. Figure 1b shows that $\mathcal{Q}_H$ generally falls somewhere between these two limits. A reasonable approximation is obtained assuming that T_2 varies linearly with latitude, so that $C_2 = H_m - 2\Delta H y/L_Y$. Plugging this into Eq. (12) then gives $\mathcal{Q}_H = \int_{-L_Y/2}^0 (H_1 - C_2) dy \approx \Delta H \{ \sigma \log[\cosh(L_Y/2\sigma)] - (L_Y/4) \}$. Note that the lack of scatter in Fig. 1b also implies that $\mathcal{Q}_H$ is essentially a function of ΔH , with very little dependence on H_m .

Figure 2 shows contours of potential temperature in this model for simulations keeping $H_m = 3 \text{ K day}^{-1}$ constant and varying ΔH . For the same vertical destabilization, the isentropic slope flattens and the criticality decreases as we increase the differential heating $\mathcal{Q}_H$. This non-intuitive behavior is a consequence of the requirement

that the eddy heat flux be aligned with the isentropic slope [cf. Eq. (9a)] and is consistent with the finding in ZV09 that the criticality decreases with increasing radiative-equilibrium baroclinicity. In the limit $\Delta H = 0$, when there are no baroclinic eddies and $\mathcal{Q}_H = 0$, the flow is convectively neutral and all the eddy vertical heat transport is due to grid-scale convection. Figure 3 shows the latitudinal profiles of $w'\theta'$ for these and other simulations with the same value of H_m . The $w'\theta'$ profile is flat in the nonbaroclinic limit ($\Delta H = 0$) but decreases poleward for larger ΔH because of the latitudinal dependence of H_1 . Superimposed to that trend, there is additional meridional structure due to the eddy heating. However, note that all the simulations have the same $w'\theta'$ at y_m (midchannel), where the divergence of the meridional heat transport vanishes, and the same domain mean, as required by the energy balance. That this remains true when ΔH is small supports our above claim that grid-scale convection provides the necessary vertical transport to satisfy the energy balance in the limit of weak

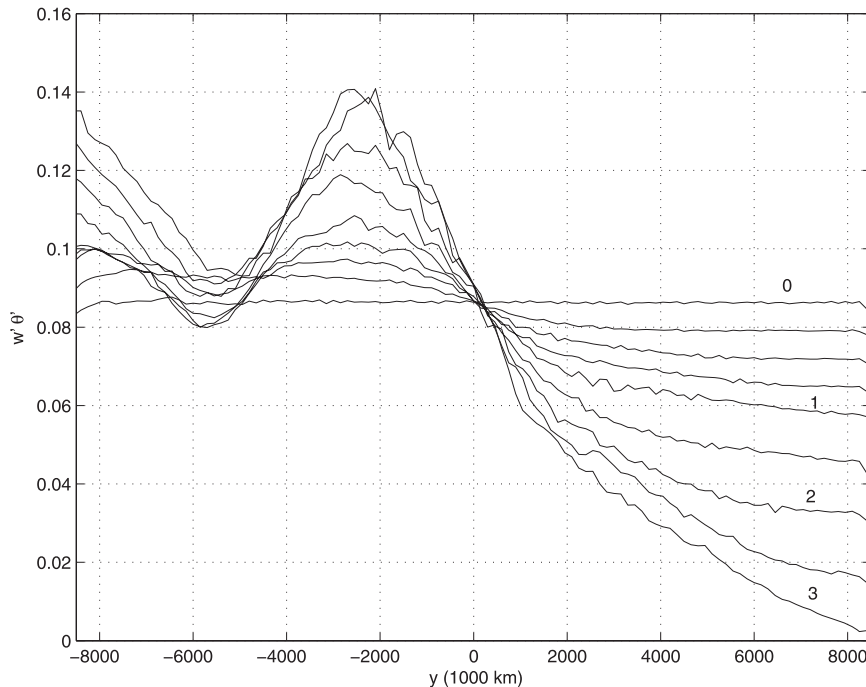


FIG. 3. Latitudinal profile of eddy vertical heat flux for simulations with fixed H_m ($H_m = 3 \text{ K day}^{-1}$) and varying ΔH . The ΔH values are indicated for some select curves.

baroclinic eddies. Figure 4 shows the sensitivity of ξ , λ , and their product on ΔH for these runs keeping constant H_m . When ΔH decreases, the criticality increases as discussed above, but this is also accompanied by a reduction in the Rossby radius, in such a manner that the $\xi\lambda$ product remains roughly constant as predicted by the theory. This is violated for small ΔH , a limit in which baroclinic eddies are weak, convective transport dominates, and the QG turbulent closure is not expected to work.

Figure 5 tests the theory more thoroughly, including other values of H_m . Figure 5a displays the dependence of $\xi\lambda$ on Q_V for all runs considered. We use for this figure the vertical destabilization Q_V diagnosed from the model, including corrections to H_m by the MMC transport and the tilt in the meridional heat flux. Each dot corresponds to a different simulation, and we use different symbols to bin the simulations in bands of the $\Delta H/H_m$ ratio (see caption for details). The theory predicts that $\xi\lambda$ should

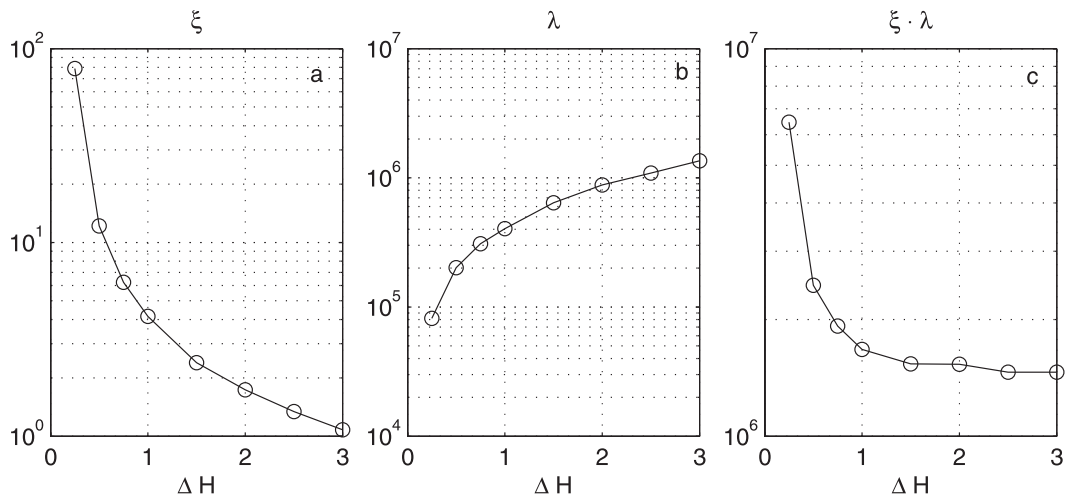


FIG. 4. For the simulations with constant $H_m = 3 \text{ K day}^{-1}$, (a) criticality ξ , (b) Rossby radius λ , and (c) their product as a function of the parameter ΔH .

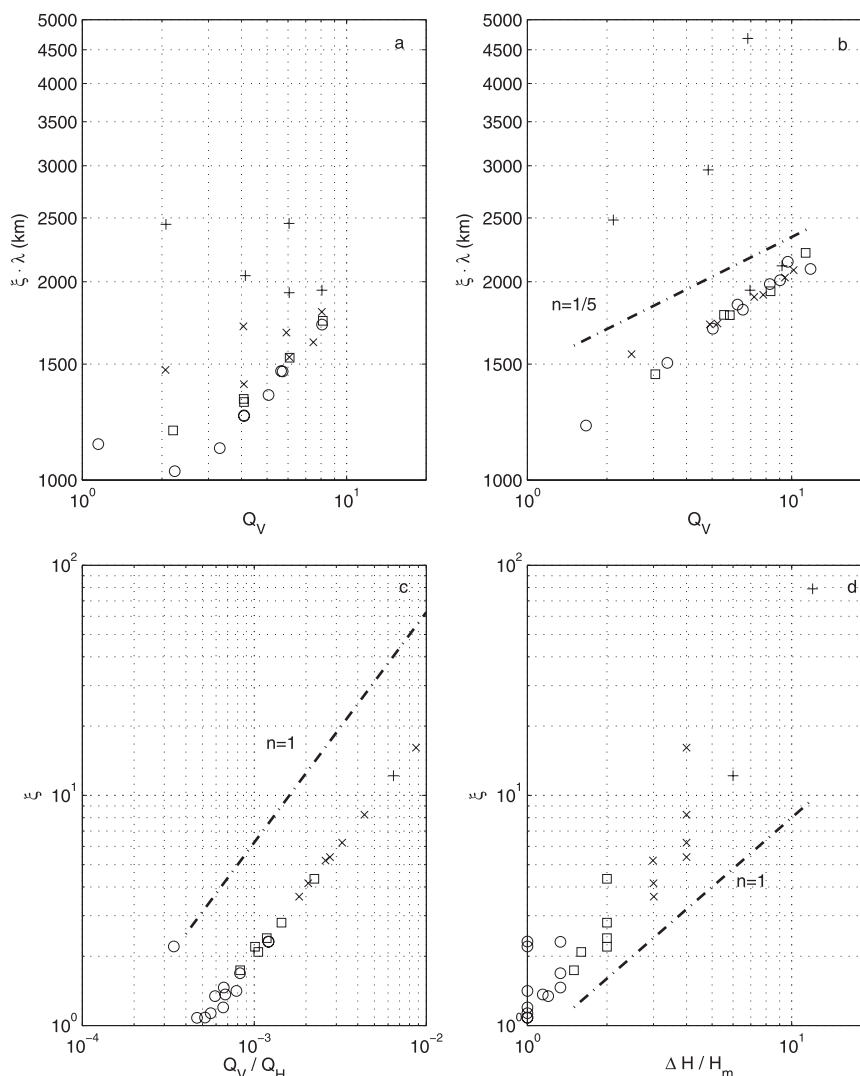


FIG. 5. (a) The $\xi\lambda$ product as a function of vertical destabilization. (b) As in (a), but using a different choice of reference latitude y_m (see text for details). (c) Criticality as a function of Q_V/Q_H . (d) As in (c), but as a function of the external parameters $H_m/\Delta H$. In all panels, each symbol denotes a different range of the ratio $\delta = \Delta H/H_m$ (pluses: $\delta \leq 0.25$, crosses: $0.25 < \delta \leq 0.5$, squares: $0.5 < \delta \leq 0.75$, and circles: $0.75 < \delta \leq 1$).

depend only on Q_V (i.e., on H_m) but not on Q_H or ΔH , apart from some possible dependence on ΔH through the corrections to H_m . In contrast, the large scatter in the empirical results suggests some dependence of $\xi\lambda$ on the $\Delta H/H_m$ ratio. However, this seems to be sensitive to the choice of reference latitude. If we define y_m as the latitude of maximum lower-level wind (as in ZV09) instead of that of maximum meridional eddy heat flux as in section 2, the scatter is largely reduced (Fig. 5b), suggesting that $\xi\lambda$ only depends on Q_V . A few simulations are outliers, but these are typically associated with small values of ΔH , a limit in which the diffusive closure is likely invalid. The slope of the empirical relation

between $\xi\lambda$ and Q_V is flat, although somewhat steeper than predicted by the theory. The implication of this weak dependence is that as long as Q_V does not change too much, it should be hard to change the length scale in a model forced from below, which might explain the insensitivity of the length scale in the simulations of Frierson et al. (2007). Figure 5c displays the sensitivity of the criticality on the Q_V/Q_H ratio, showing good agreement (this is true with both y_m choices) with the theoretical prediction. Note that this prediction is really diagnostic because although $Q_V \approx H_m$, Q_H is internally determined in the model and had to be diagnosed from the model output to construct the figure. To build a fully closed

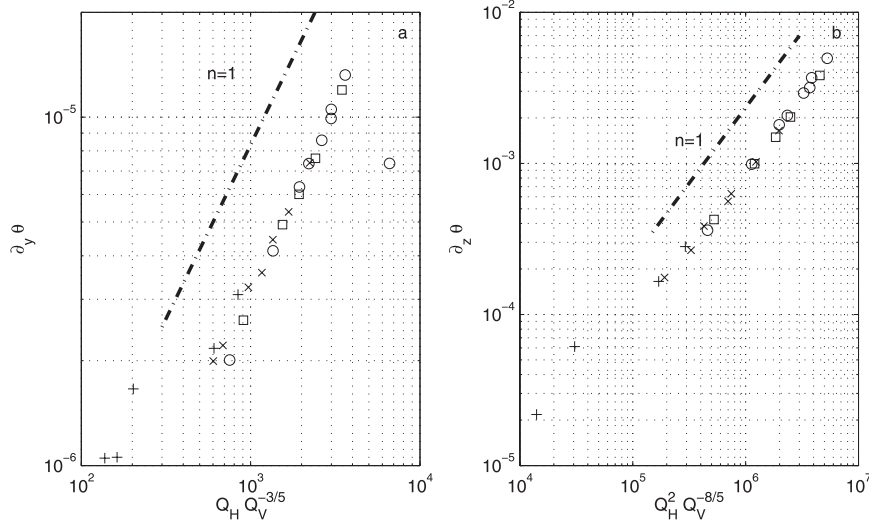


FIG. 6. (a) Meridional and (b) vertical time-mean temperature gradients in the model, compared to their predicted dependence on the heating. Symbols denote different ranges of the $\Delta H/H_m$ ratio as in Fig. 5.

theory, one could assume that Q_H is proportional to ΔH (cf. Fig. 1b), which still retains some predictive skill (Fig. 5d). Finally, Fig. 6 shows that the theoretical predictions for the meridional and vertical temperature gradients also work well once the heating is known.

4. Newtonian forcing model

In this section we revisit the Newtonian cooling problem discussed in ZV09, using the same simulations presented there. The results of that paper imply that the diffusive closure works reasonably well, which suggests that the constraints put forward in section 2 should also be relevant for this problem. However, the sensitivity of the mean state on the external parameters is harder to understand than in the prescribed heating model because the heating is now fully internally determined.

As described in more detail in ZV09, this model is forced by linear relaxation to a prescribed radiative-equilibrium profile of the form

$$\theta_R = \Theta_0 + \frac{\delta_Z}{2} \times \delta_{k,2} - \frac{\delta_Y}{2} \tanh\left[\frac{y - (L_Y/2)}{\sigma}\right] \quad k = 1, 2, \quad (13)$$

where L_Y and σ are the same as in the prescribed heating model. The key forcing parameters with this heating formulation are the radiative-equilibrium baroclinicity δ_Y , the radiative equilibrium stratification δ_Z , and the forcing time scale τ . To increase the baroclinic forcing one may change either δ_Y or τ . However, in contrast with the previous model, changes in these parameters affect

the vertical destabilization too because the heating is internally determined.

Figure 7, adapted from ZV09, displays the sensitivity of the criticality on the forcing parameters for this model. The left panel shows the sensitivity varying δ_Y and δ_Z while keeping τ at its control value. We can see that the criticality decreases in general with increasing δ_Y , except when δ_Y is small and δ_Z is large. This sensitivity is similar to that displayed by the prescribed heating model, but opposite to that displayed by the quasigeostrophic model when changing the same parameter. Note that there are also regions with horizontal criticality isolines, implying weak sensitivity to δ_Y : this happens, for instance, around the control values $\delta_Y = 60$ K, $\delta_Z = 40$ K in ZV09. Figure 7b shows the sensitivity when τ is varied, now keeping δ_Z constant at its control value. We can see that the criticality always increases with decreasing τ . As long as τ is not too small, the criticality decreases with increasing δ_Y , as already noted above for its control value $\tau = 20$ days. However, when τ becomes very small the sensitivity is reversed and the criticality increases with increasing δ_Y .

To understand this complex sensitivity, we need to understand how the heating changes with the external parameters. The key point is that changes in δ_Y affect *both* the differential heating Q_H and the vertical destabilization Q_V because the latter is now internally determined. Hence, while we would expect ξ to decrease with increasing Q_H as in the prescribed heating model, this should be tempered by the concurrent changes in Q_V , which also tends to increase with increasing baroclinicity. Indeed, if the vertical destabilization were to

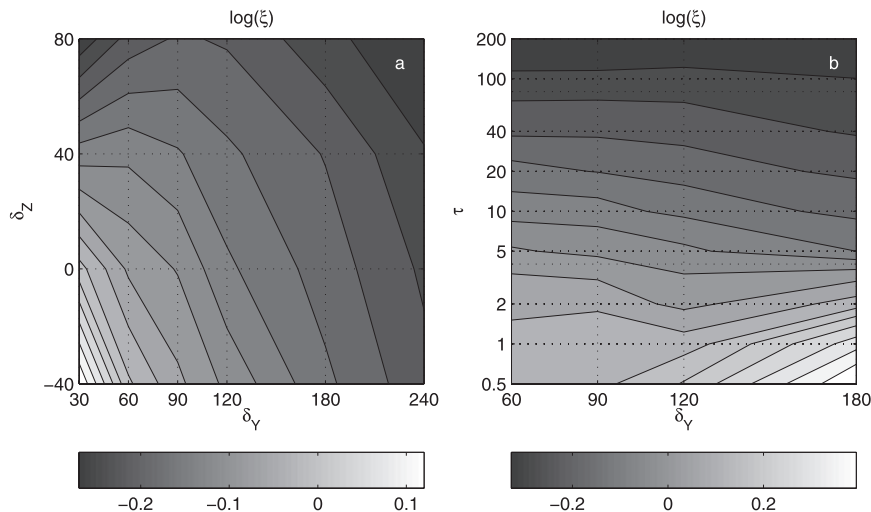


FIG. 7. Logarithm of criticality in the Newtonian cooling model when (a) varying δ_Y and δ_Z keeping $\tau = 20$ days constant and (b) varying δ_Y and τ keeping $\delta_Z = 40$ K constant. Light (dark) shades correspond to high (low) criticalities. Adapted from ZV09.

increase more than the differential heating, then the criticality would decrease [cf. Eq. (9a)].

To be more quantitative, consider the definition of the vertical destabilization in this model:

$$Q_V \sim \frac{\partial_z \bar{\theta} - \delta_Z}{\tau} \propto \frac{(\partial_z \bar{\theta})^r}{\tau}. \quad (14)$$

For a neutrally stratified radiative equilibrium profile ($\delta_Z = 0$), $r = 1$, a case already discussed by Held (2007). The quasigeostrophic problem is obtained in the limit $r \rightarrow \infty$ (in that limit Q_V changes at constant $\partial_z \bar{\theta}$). More generally, the appendix shows that one may approximate

$$\partial_z \bar{\theta} - \delta_Z \approx \frac{\tilde{\theta}_z - \delta_Z}{\tilde{\theta}_z^r} (\partial_z \bar{\theta})^r \propto (\partial_z \bar{\theta})^r, \quad (15)$$

where $r = 1/(1 - \kappa)$, $\kappa = \delta_Z/\tilde{\theta}_z$, and $\tilde{\theta}_z$ is the characteristic stratification of the mean state about which the sensitivity is examined. Note that for $\delta_Z > 0$ this implies a steeper than linear dependence of the heating on the stratification, and that Q_V changes quite rapidly with stratification when the stratification is close to its radiative equilibrium value ($r \rightarrow \infty$ for $\tilde{\theta}_z \rightarrow \delta_Z$), which corresponds to the quasigeostrophic limit.

Plugging $\partial_z \bar{\theta} \sim \tau^{1/r} Q_V^{1/r} = \tau^{1-\kappa} Q_V^{1-\kappa}$ into Eq. (10b) and clearing Q_V yields

$$Q_V \sim \tau^{(5\kappa-5)/(13-5\kappa)} Q_H^{10/(13-5\kappa)}, \quad (16)$$

where constant factors have been omitted for clarity. We can see that Q_V always changes when Q_H changes, which

implies that this model's sensitivity to differential heating should be different from the prescribed heating model. Indeed, plugging this into Eq. (9a), we obtain

$$\xi \sim \frac{Q_V}{Q_H} \sim \tau^{(5\kappa-5)/(13-5\kappa)} Q_H^{(5\kappa-3)/(13-5\kappa)}. \quad (17)$$

Figure 8a shows the dependence of the Q_H exponent, n , as a function of the parameter $\kappa = \delta_Z/\tilde{\theta}_z$. Because, the eddies always enhance the stratification relative to radiative equilibrium, κ must be bounded by 1. On the other hand, κ is negative when relaxing to a convectively unstable profile ($\delta_Z < 0$), since $\tilde{\theta}_z$ must be nonnegative. Moreover, one can have $\kappa \rightarrow -\infty$ when relaxing to a very unstable profile and/or when the equilibrated state is marginally neutral. For these values of κ , Fig. 8a shows that the sensitivity is bounded by $-1 \leq n \leq 1/4$. The upper bound corresponds to the quasigeostrophic limit, reached for $\kappa = 1$ or $\tilde{\theta}_z = \delta_Z$. The lower bound is reached asymptotically as $\kappa \rightarrow -\infty$ (or $r \rightarrow 0$), a limit in which we recover the sensitivity for the prescribed heating model ($n = -1$), corresponding to changes in criticality at constant Q_V .

It is interesting that the sign of the sensitivity changes depending on the value of κ . For $\kappa > 0.6$ (shaded region), the criticality increases with increasing differential heating as for the quasigeostrophic model, albeit with a weaker sensitivity $n \leq 1/4$. In contrast, the criticality decreases with increasing differential heating when $\kappa < 0.6$, as is always the case when relaxing to a convectively unstable or neutral radiative-equilibrium profile ($\kappa \leq 0$). Moreover, note that for convectively unstable profiles the sensitivity is typically larger (albeit opposite in sign)

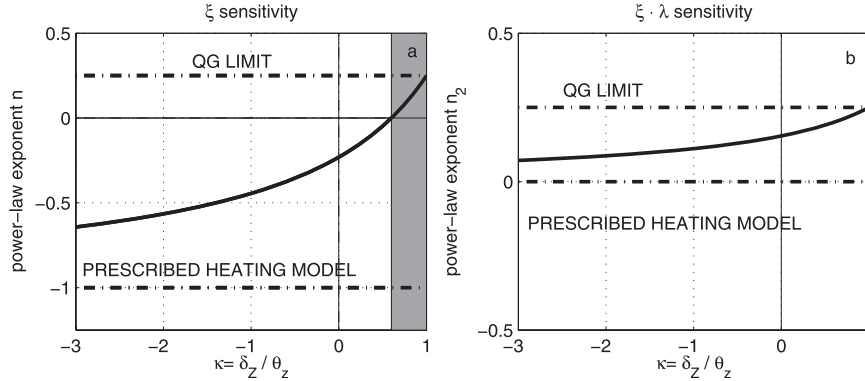


FIG. 8. Exponents n and n_2 measuring the Q_H dependence in the analytical closures for (a) ξ [cf. Eq. (17)] and (b) $\xi \cdot \lambda$ [cf. Eq. (18)], respectively, as a function of δ_Z / θ_z . The dashed-dotted lines show the QG limit and the sensitivity for the prescribed heating model. The shaded area in (a) emphasizes the region of parameter space with positive sensitivity.

than the quasigeostrophic sensitivity, since $n = -3/13 \approx -0.23$ already in the neutral case. Finally, the criticality becomes essentially insensitive to differential heating when $\delta_Z \sim O(0.6 \times \tilde{\theta}_z)$.

These predictions are tested in Fig. 9a, which shows the criticality as a function of the *diagnosed* differential heating varying δ_Y (Q_H increases monotonically with δ_Y in this model). This is done for four different values of the radiative equilibrium stratification δ_Z , using the same simulations included in Fig. 6b of ZV09. This includes the control stratification ($\delta_Z = 40$ K), a highly stratified radiative equilibrium profile ($\delta_Z = 80$ K), a convectively unstable radiative equilibrium ($\delta_Z = -40$ K), and a neutral radiative equilibrium ($\delta_Z = 0$ K). With

convectively neutral or unstable radiative equilibria, $\kappa = \delta_Z / \tilde{\theta}_z < 0.6$ always, and the criticality decreases monotonically with increasing differential heating. It does so with a slope that is $O(1/4)$ in the neutral case and only slightly steeper in the unstable case. In contrast, for positive δ_Z the sensitivity is nonmonotonic and the criticality increases (decreases) with increasing differential heating when Q_H is weak (strong). Both growth and decay exhibit a flat slope: $|n| < 1/4$. The criticality has a maximum at intermediate values of Q_H , such that $\delta_Z \approx 0.6 \times \tilde{\theta}_z$. This transition occurs because $\tilde{\theta}_z$, which is equal to δ_Z for $Q_H = 0$, increases monotonically with Q_H , implying that $\delta_Z / \tilde{\theta}_z$ must decrease monotonically from 1 as Q_H increases from zero, eventually going through $\kappa = 0.6$

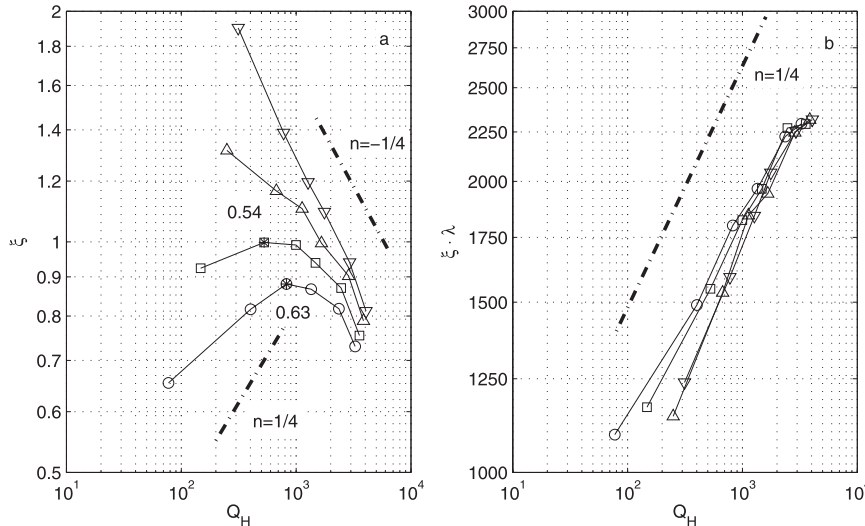


FIG. 9. (a) Empirical criticality in the Newtonian cooling model as a function of differential heating, in four sets of runs varying δ_Y with fixed δ_Z : $\delta_Z = 80$ (circles), 40 (squares), 0 (triangles), and -40 K (inverted triangles). We also indicate the value of $\delta_Z / \tilde{\theta}_z$ at the point of maximum criticality when a local maximum exists. The theory predicts that this maximum should occur at $\kappa = 0.6$. (b) As in (a), but for the $\xi \lambda$ product.

when Q_H is large enough. All these features are in accord with the theoretical analysis.

The above analysis also allows us to understand why the sensitivity of the criticality on radiative-equilibrium baroclinicity is opposite for large and small τ (cf. Fig. 7b). While we expect the criticality to decrease with increasing δ_Y with realistic values of τ , this tendency has to be reversed for small τ . The reason is that when τ becomes very small, the flow must approach radiative equilibrium and ξ_R increases with increasing δ_Y (for the same positive δ_Z). This is still consistent with the above analysis because for small τ the stratification approaches δ_Z and κ exceeds 0.6. As $\kappa \rightarrow 1$ we recover the sensitivity of the quasigeostrophic limit.

It is also of interest to analyze the sensitivity of $\xi\lambda$ [an estimate for the eddy length scale in the theory of Held and Larichev (1996)] on differential heating for the Newtonian cooling model. As we saw, this product was insensitive to Q_H in the prescribed heating model. However, the situation is different for the Newtonian cooling model because Q_V now changes with Q_H :

$$\xi\lambda \sim Q_V^{1/5} \sim \tau^{(\kappa-1)/(13-5\kappa)} Q_H^{2/(13-5\kappa)}. \quad (18)$$

The Q_H exponent, n_2 , is plotted in Fig. 8b, together with the predictions for the quasigeostrophic model (for which $n_2 = n = 1/4$, since λ is fixed) and the prescribed heating model ($n_2 = 0$, no sensitivity). The sensitivity is positive definite, implying that the length scale and the energy level should always increase with differential heating, albeit at a slightly slower rate than for the quasigeostrophic model. This is roughly consistent with the numerical model results (Fig. 9b), which show that $\xi\lambda$ increases monotonically with Q_H , with an exponent which is actually fairly constant and close to the quasigeostrophic prediction $n_2 = 1/4$ (flattening, if any, only becomes apparent for the largest values of Q_H). This implies that in spite of the very different sensitivity of ξ reported in Fig. 9a, the length scale $L \sim \xi\lambda$ and the strength of the energy cycle ($\epsilon \sim (g/\Theta_0)\overline{w'\theta'} \sim \beta^3 L^5$) should actually exhibit a similar sensitivity on differential heating in all cases, a sensitivity also comparable to that of the quasigeostrophic model.

To conclude, we discuss the *global* sensitivity of ξ and $\xi\lambda$ on Q_H , as opposed to the *local* sensitivity around specific values of θ_z considered so far. Although the exponents n and n_2 have a comparable order of magnitude, the exponent n is more variable and can actually change sign as Q_H is varied, provided that $\delta_Z > 0$. This leads to a nonmonotonic sensitivity of the criticality on differential heating and to smaller relative differences overall in ξ than in $\xi\lambda$ when changing Q_H over a finite range, as shown in Fig. 9. In contrast, n and n_2 are equal and constant for

the quasigeostrophic model, so that ξ and $\xi\lambda$ display the same weak but uniform $1/4$ sensitivity in that case. The implication is that when $\delta_Z > 0$, the baroclinic adjustment paradigm of a constant ξ should work even better in the primitive equation than in the quasigeostrophic model, even though the eddy scales display the same weak sensitivity on the forcing in both cases.

Sensitivity to diabatic time scale

Finally, we discuss the sensitivity of the Newtonian cooling model on the diabatic time scale. We use for illustration the sets of simulations with $(\delta_Y, \delta_Z) = (60, 40)$, and $(60, 0)$ and varying τ , described in ZV09. The results are shown in Fig. 10.

When the forcing time scale τ is reduced, we expect both the differential heating Q_H and the vertical destabilization Q_V to increase, albeit with a weak (sublinear) dependence based on the arguments³ of Zurita-Gotor and Lindzen (2006). This is confirmed by the numerical results shown in Figs. 10e,f. Additionally, we expect Q_V to increase faster than Q_H since the criticality

$$\xi(\tau) \propto \frac{Q_V(\tau)}{Q_H(\tau)}$$

should increase with decreasing τ as the flow approaches radiative equilibrium. This prediction is well satisfied in Fig. 10 for moderate and large τ , but not when τ is small. In that limit, ξ increases with decreasing Q_V/Q_H and the above scaling fails, presumably because the adiabatic eddy assumption is violated and the mixing slope is flatter than the isentropic slope. Given the restoring nature of both heating terms in the Newtonian formulation, the implication of the Q_V/Q_H sensitivity on τ is that the convergence to radiative equilibrium should occur faster in the meridional than in the vertical direction as τ is decreased. This is supported by Fig. 6c of ZV09 and, for the multilevel case, by Fig. 4b of Zurita-Gotor (2008), as well as by our Figs. 10c,d (although the $\Delta_r\theta$ variations are small in this figure). We can see in all these figures that the meridional temperature gradient saturates first, when the stratification is still far from radiative equilibrium. Further increases in criticality beyond this point are due to reductions in $\Delta_r\theta$, as the stratification converges to radiative equilibrium. The slope of the criticality dependence is somewhat different before and after

³ Their arguments go as follows: When the forcing time scale τ decreases, Q must increase because the circulation strengthens, but $T - T_r \sim \tau Q$ must decrease because the flow is closer to radiative equilibrium. Both conditions combined imply that $Q \sim 1/\tau^n$ with $0 < n < 1$.

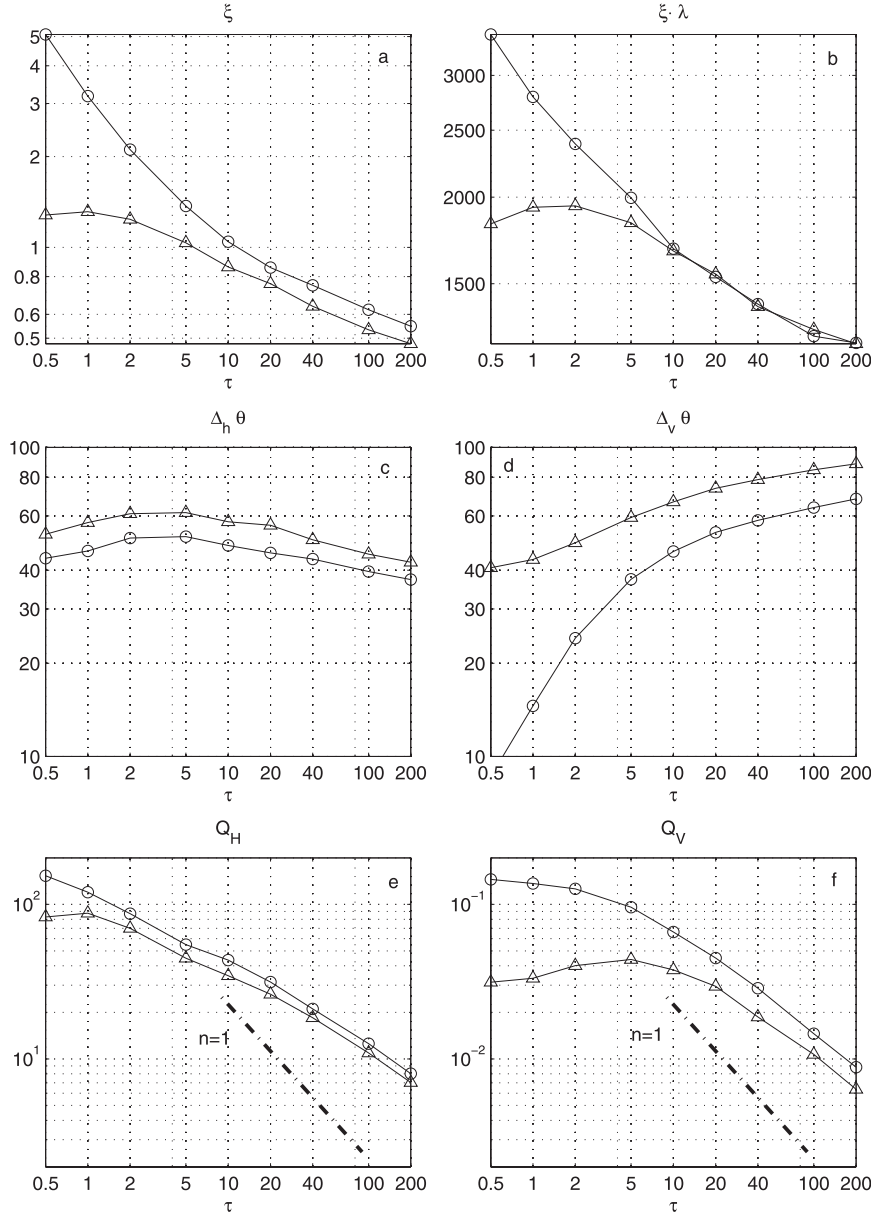


FIG. 10. Sensitivity of (a) ξ , (b) $\xi\lambda$ product, (c) meridional temperature gradient, (d) vertical temperature gradient, (e) differential heating Q_H , and (f) vertical destabilization Q_V on the forcing time scale for sets of simulations with $\delta_Y = 60$ K and $\delta_Z = 40$ K (triangles) or $\delta_Z = 0$ K (circles).

$\Delta_h\theta$ saturates, but the overall sensitivity is smooth and monotonic.

A final question of interest is whether the sensitivity on the forcing time scale is stronger or weaker for the primitive equation model compared to a quasigeostrophic model. To investigate this issue, we return to Eqs. (17) and (18):

$$\xi \sim \tau^{(5\kappa-5)/(13-5\kappa)} Q_H^{(5\kappa-3)/(13-5\kappa)} \quad \xi\lambda \sim \tau^{(\kappa-1)/(13-5\kappa)} Q_H^{2/(13-5\kappa)}.$$

As noted above, one can further approximate $Q_H \sim \tau^{-\gamma}$ with $0 \leq \gamma \leq 1$. Figure 10e shows that γ is reasonably uniform and independent of the stratification in our runs, for which the best fit is obtained with $\gamma = 0.5$. Figure 11 shows the resulting sensitivities for some plausible values of γ . We can see that the sensitivity is negative definite, implying that both ξ and $\xi\lambda$ increase monotonically with decreasing τ , though the latter is much less variable. Assuming that γ does not change, the sensitivity increases

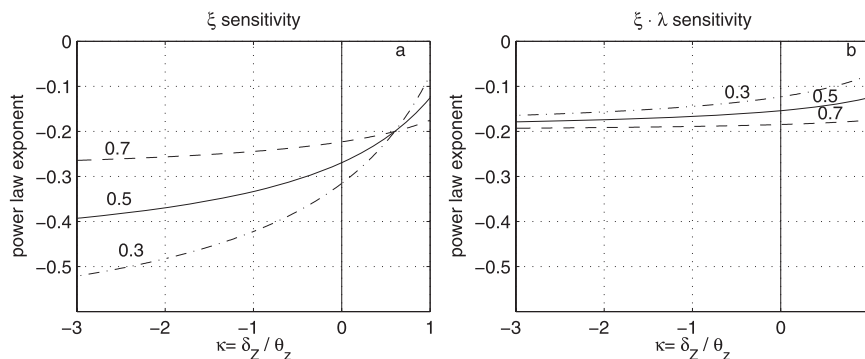


FIG. 11. Exponents measuring the dependence on τ of the analytical closures for (a) ξ and (b) $\xi\lambda$ as a function of $\kappa = \delta_z/\theta_z$ and for different values of γ , as indicated. The quasigeostrophic problem is obtained for $\kappa = 1$.

as κ decreases, and is minimum in the quasigeostrophic limit $\kappa = 1$.

5. Concluding remarks

In the theory of Held and Larichev (1996) the eddy length scale scales as $L \sim \xi\lambda$, the velocity scale as $V \sim \beta L^2$, the diffusivity as $D \sim VL \sim \beta L^3$, and the strength of the energy cycle as $\epsilon \sim (g/\Theta_0)w'\theta' \sim \beta^3 L^5$. As the forcing is increased, we expect the energy level to increase and hence expect a larger $\xi\lambda$ product. However, this does not fully constrain the criticality ξ except in particular cases. In the quasigeostrophic limit, the Rossby radius is fixed and all the increase in the length scale arises through supercriticality and an inverse cascade. In contrast, in the baroclinic adjustment limit the criticality ξ stays constant, the flow remains linear, and eddy lengths scale as the deformation radius. However, these two limits only represent two among many other possible state evolutions as the forcing is varied.

The scaling $\xi \sim Q_V/Q_H$ implies that, in fact, the criticality should be better regarded as dependent on the structure of the heating rather than on its strength. Although this does not tell us the full answer because the heating is in general internally determined, it allows us to understand the different sensitivity on differential heating noted in previous studies using different formulations. This range of sensitivities is due to the fact that Q_V may also change with Q_H : when it increases faster (slower) than Q_H , the criticality increases (decreases). The outcome ultimately depends on the steepness of the vertical heating relation $Q_V(\partial_z \theta) \sim (\partial_z \theta)^r$, which is both model dependent and state dependent for the same model. The quasigeostrophic problem ($r = \infty$) and the prescribed heating model ($r = 0$) constrain the plausible range of behaviors and provide bounds of sensitivity for our model.

Although the prescribed heating model introduced in this study may seem artificial, it shares some similarities with the more complex radiative codes used in continuous models. In particular, the net atmospheric vertical heat transport from the surface to cooling levels is also strongly constrained in equilibrium by the incoming solar radiation in the gray radiation model, though not exactly fixed because infrared radiative feedbacks also play a role as temperatures change in that case. What makes our heating formulation most unrealistic is not so much the prescription of the net vertical transport but rather the specification of a unique, fixed depth over which this vertical transport must occur. In reality this depth is not only internally determined but also nonunique and different for all processes, with both shallow and deep convection potentially playing a role.

Frierson et al. (2006) recently developed an idealized moist general circulation model forced with a gray radiation scheme and Frierson et al. (2007) studied the sensitivity of the energy transports in that model on moisture. A major result of this work was that the (dry) isentropic slope in the moist GCM flattens as moisture increases, in contrast with the remarkable robustness of the criticality in dry models forced by Newtonian cooling (Schneider 2004). The same seems to be true in the moist model of Schneider and O'Gorman (2008), even though its dry limit is qualitatively different from that of Frierson et al. A possible reading of these results is that dry and moist models are fundamentally different. However, the model of Frierson et al. (2006) displays sensitivity to moisture even at low moisture values, when moist effects should be relatively unimportant. Moreover, even in the dry limit, the mean state in that model differs significantly from that in the Newtonian cooling model and from the dry limit of Schneider and O'Gorman (2008), obtained using a different methodology. This suggests that the different internal determination of the heating in all these models

could also contribute to their different behavior, apart from any influence that moisture may likely have. The impact of moisture on the extratropical equilibrium is a very difficult and poorly understood question, deserving of further investigation. While we obviously cannot account for the effects of moisture using a dry framework, it is worth asking whether the sensitivity of the criticality on moisture reported in previous studies is at least consistent with the expected changes in the structure of the heating as moisture is added.

Neglecting the effect of moisture on the eddies (the $\overline{Q'\theta'}$ term), Eq. (4) should still hold in the moist case. To the extent that the turbulence closures can be extrapolated to the multilevel case then, the dry framework developed here remains valid and moisture only enters the problem through the latent contributions to Q_H and Q_V . Essentially, the meridional and vertical latent transports by baroclinic eddies and convection tend to reduce the effective differential heating and vertical destabilization seen by the dry problem relative to the full radiative forcing. When Q_V reduces more than Q_H with moisture (i.e., the net vertical latent transport, including convection, increases more than the latent meridional transport), the dry theory predicts a flattening of the isentropes with moisture. This is what should be expected because as moisture increases the convective transport becomes more important.

This is of course pure speculation, as our dry two-level model is too idealized and there are many caveats that could invalidate the above arguments. First of all, the effect of moisture on the eddies is probably not negligible because the condensational generation of eddy available potential energy is thought to fuel the eddies even in the present climate and should play more of a role at high moisture. Additionally, the use of a two-level discretization and, relatedly, a fixed tropopause height is a major simplification that could invalidate our results, even in the dry case. In a more realistic model the vertical scale of the heating is also internally determined, and this may affect the partition between differential heating and vertical destabilization and thus the criticality. It is plausible that this could attenuate the changes in criticality relative to our two-level model, although the results of Chang (2006) suggest that one can also obtain any desired criticality in that problem provided that the right heating is used. We are currently investigating whether adjustment to marginal criticality is robust against changes in the heating in dry models that determine their own tropopause.

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APPENDIX

Derivation of Eq. (15)

To study the sensitivity of the criticality on the heating with the Newtonian cooling formulation, it is convenient to write the vertical destabilization as a power law or to approximate

$$\partial_z \bar{\theta} - \delta_Z \sim C(\partial_z \bar{\theta})^r. \quad (\text{A1})$$

We achieve this by Taylor-expanding $\log(\partial_z \bar{\theta} - \delta_Z)$ in powers of $\log(\partial_z \bar{\theta})$. We expand about some characteristic mean state with stratification $\partial_z \bar{\theta} = \tilde{\theta}_z$:

$$\begin{aligned} \log(\partial_z \bar{\theta} - \delta_Z) &\approx \log(\tilde{\theta}_z - \delta_Z) + \frac{\partial \log(\partial_z \bar{\theta} - \delta_Z)}{\partial \log(\partial_z \bar{\theta})} \bigg|_{\tilde{\theta}_z} \\ &\quad \times [\log(\partial_z \bar{\theta}) - \log(\tilde{\theta}_z)] + \dots \\ &\approx \log(\tilde{\theta}_z - \delta_Z) + \frac{\tilde{\theta}_z}{\tilde{\theta}_z - \delta_Z} \log\left(\frac{\partial_z \bar{\theta}}{\tilde{\theta}_z}\right) + \dots \end{aligned} \quad (\text{A2})$$

Using the exponential function to get rid of the logs, we obtain

$$\partial_z \bar{\theta} - \delta_Z \approx \frac{\tilde{\theta}_z - \delta_Z}{\tilde{\theta}_z^r} (\partial_z \bar{\theta})^r, \quad (\text{A3})$$

where $r = \tilde{\theta}_z / (\tilde{\theta}_z - \delta_Z)$. Note that $\tilde{\theta}_z$ and r on the right-hand side are constants after the expansion so that, to first order, all the sensitivity on the stratification arises through the $(\partial_z \bar{\theta})^r$ term when the variations in $\partial_z \bar{\theta}$ with respect to its characteristic value $\tilde{\theta}_z$ are small.

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