

## Equilibration of Baroclinic Turbulence in Primitive Equations and Quasigeostrophic Models

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### ABSTRACT

This paper investigates the equilibration of baroclinic turbulence in an idealized, primitive equation, two-level model, focusing on the relation with the phenomenology of quasigeostrophic turbulence theory. Simulations with a comparable two-layer quasigeostrophic model are presented for comparison, with the deformation radius in the quasigeostrophic model being set using the stratification from the primitive equation model. Over a fairly broad parameter range, the primitive equation and quasigeostrophic results are in qualitative and, to some degree, quantitative agreement and are consistent with the phenomenology of geostrophic turbulence.

The scale, amplitude, and baroclinicity of the eddies and the degree of baroclinic instability of the mean flow all vary fairly smoothly with the imposed parameters; both models are able, in some parameter ranges, to produce supercritical flows. The criticality in the primitive equation model, which does not have any convective parameterization scheme, is fairly sensitive to the external parameters, most notably the planet size (i.e., the  $f/\beta$  ratio), the forcing time scale, and the factors influencing the stratification. In some parameter settings of the models, although not those that are most realistic for the earth's atmosphere, it is possible to produce eddies that are considerably larger than the deformation scales and an inverse cascade in the barotropic flow with a  $-5/3$  spectrum. The vertical flux of heat is found to be related to the isentropic slope.

### 1. Introduction

Understanding the mechanisms of equilibration of baroclinic eddies in the atmosphere and ocean, and indeed in stratified rotating fluids more generally, remains an outstanding problem in the atmospheric and oceanic sciences. The problem is ultimately one of turbulence, albeit not three-dimensional turbulence at small scales, as encountered for example in boundary layers and in laboratory settings. Rather, the flow occurs at “large” scales, meaning at scales similar to the first radius of deformation, which is about 1000 km in the atmosphere and 100 km in the ocean. This kind of turbulence is variously known as baroclinic turbulence, because of its connection with baroclinic instability theory, geostrophic turbulence, because the flow is nearly geostrophically

balanced, or macroturbulence, reflecting the large-scale nature of the flow. This kind of turbulence need not be, and at least in the earth's atmosphere likely is not, fully developed turbulence in any way: it is intermingled with Rossby waves, and the turbulent cascades that characterize classical turbulence (in both two and three dimensions) are by no means well formed in the earth's atmosphere. Nevertheless, the flow is certainly nonlinear and to some degree both spatially and temporally disordered. In this paper we explore the mechanisms of equilibration of such turbulence, and in particular the processes that give rise to the amplitude and scale of the flow, using both primitive equation (PE) and quasigeostrophic (QG) models.

Although linear baroclinic instability theory is well established, the relevance, or even the existence, of modal development or exponential growth for the eddy-mean flow interaction problem is unclear, and the equilibration of baroclinic eddies at finite amplitude remains a complex problem. An early attempt to address the nonlinear

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problem was that of Green (1970), who attempted to build a theory of the general circulation based on the diffusion of potential vorticity (PV) by baroclinic eddies. Given that the dimensions of eddy diffusivity are those of length times velocity, he suggested that the length scale be that of the entire baroclinic zone and that the velocity scale be determined by energetic considerations based on the amount of available potential energy in the mean flow. The equilibration of the mean flow would then be determined (using potential vorticity diffusion as a framework) by a balance between the extraction of energy by eddies of this scale and magnitude and the radiative forcing on the system.

Without attempting to address the magnitude or scale to which eddies would grow, Stone (1978) argued that the effects of baroclinic eddies would be akin to that of convection on stratification. That is, the baroclinic eddies would be so efficient in transferring energy poleward and upward that they would effectively stabilize the flow to further instances of baroclinic instability, a process known as baroclinic adjustment. The upshot would be a mean flow marginally neutral to baroclinic instability, at least in some sense, after a zonal and time averaging. Continuously stratified flow generally has no critical shear, but if the shear is greater than the critical shear of the two-layer problem then the poleward heat transport will likely be more efficient (Held 1978). The method whereby the flow putatively becomes marginally neutral is not specified by Stone's idea and might involve either poleward transport of heat, reducing the meridional temperature gradient and thus the vertical shear of the zonal wind, or upward heat transport, increasing the static stability and concomitantly increasing the deformation radius and reducing the supercriticality of the flow. Similarly, and more recently, Schneider (2004) and Schneider and Walker (2006) studied the behavior of the criticality in a multilevel, primitive equation model on the sphere like that described by Held and Suarez (1994), except that a statically unstable equilibrium profile is used in conjunction with a simplified convective scheme. Schneider and Walker found that when eddies control the stratification, their model equilibrates in states of marginal criticality for many climate perturbations, more or less consistently with baroclinic adjustment theories. In this context, marginal criticality refers to a mean state with a shear that varies with parameters in a similar way to the critical shear of the two-layer model but that is unstable in general (neutrality at this particular shear is specific to the two-layer model). Schneider did not attribute this result to adjustment-to-neutrality processes; rather, he regarded it as a consequence of dynamical constraints on the mass and momentum balances, combined with an

assumption about the vertical homogeneity of the eddy diffusivity of potential vorticity in the interior and buoyancy at the boundary.

However, the reduction of the mean flow to a marginally supercritical state is certainly not the only means whereby a flow can, in general, equilibrate; for example, Salmon (1980) and Vallis (1988) both found supercritical states in turbulent quasigeostrophic flows. Rhines (1977) and Salmon (1978, 1980), adapting and building on the work of Charney (1971), proposed a conceptual model for two-layer geostrophic turbulence essentially as follows: Energy is first provided to the large-scale baroclinic flow by the external forcing (i.e., by the large-scale radiative forcing in the case of the earth's atmosphere). At large scales the temperature acts approximately as a passive tracer, so the energy in the baroclinic flow is cascaded to smaller scales until, at the scale of the first radius of deformation, it is transferred to barotropic flow. The barotropic energy then cascades upscale, creating barotropic energy at very large scales, and the general passage of energy from baroclinic to barotropic flow as part of a generalized inverse cascade has become known as barotropization. Such a conceptual model is broadly consistent with the life cycle studies of Simmons and Hoskins (1978), who generally found a tendency toward barotropic end states in their initial value, primitive equation calculations.

Larichev and Held (1995) and Held and Larichev (1996) subsequently attempted to quantify the Salmon–Rhines picture, making specific predictions for the scale and magnitude of the turbulence as a function of the various parameters of the flow, such as the value of  $\beta$ , the mean shear, and the deformation radius. Notably, the presence of an inverse cascade of barotropic energy implies that the energy-containing scale can be much larger than the deformation scale. The scaling theory also predicts that both the eddy amplitude and scale will increase quite rapidly with the mean shear, or more specifically with the criticality  $\xi = U/\beta\lambda^2$ , where  $U$  is a zonal velocity difference between the upper and lower troposphere and  $\lambda$  is the deformation radius, implying that any eddy diffusivity that might be constructed from these quantities would also increase rapidly with the shear. Note that  $\xi$ , significantly, is also a nondimensional measure of the isentropic slope.

One limitation of quasigeostrophic theory is that it takes the stratification as given, yet the extratropical stratification is also affected by the very eddy fluxes that the theory aims to model (Stone 1972), as well as by smaller-scale convective processes that are outside of the quasigeostrophic framework (Juckes 2000). Regarding the large-scale eddy fluxes, Stone proposed a simplified radiative–dynamical framework in which the diabatic

heating is balanced by the eddy heat fluxes, both meridional and vertical. The structure of these fluxes was estimated in that paper using Eady normal solutions, which implies, if the eddies are adiabatic, that the ratio between vertical and horizontal fluxes scales as the isentropic slope. This framework may be exploited to give a plausible closure for the vertical eddy heat flux: as noted by Held (2007), the vertical flux can then be calculated from the meridional heat flux—already predicted by the QG theory—and the isentropic slope. An implicit assumption is that the eddy vertical heat flux, although important for determining the stratification, does not alter the underlying quasigeostrophic theory; that is, one can still apply that theory once the stratification is known.

It is not clear that such an approach is really justifiable, because the vertical transfer of heat and the associated increase in deformation radius, omitted in quasigeostrophic theory, appear to be an important equilibration mechanism in primitive equation life cycle studies (Gutowski 1985; Gutowski et al. 1989). The implications for the forced–dissipative equilibrium were further explored by Zhou and Stone (1993a,b), who compared results from two-level simulations on the sphere with fixed and variable static stability, and both of them with previous two-layer QG results by Stone and Branscome (1992). The authors noted a tendency for the isentropic slope/criticality to remain robust against changes in the forcing in all models, albeit with a different value of  $\xi$  for each. In contrast, the meridional temperature gradient displayed much less sensitivity on the forcing in the model with fixed static stability than in the model with variable static stability.

An important process in the marginal-criticality models is the ability of the eddies to modify the stratification and/or tropopause height, effects that are unavailable in the quasigeostrophic theory. Certainly, standard turbulence models allow supercritical flows, the possibility of an extended inverse cascade, and a barotropically dominated large scale. These features are *not* unambiguously seen in the earth's atmosphere, broadly consistent with marginal criticality: a corollary of that idea is that the flow cannot support an inverse cascade—the eddy scale is just the deformation scale as in linear instability theory. However, in the ocean the eddies do seem to be significantly larger than the instability scale and, perhaps more tellingly, do not have the same geographical distribution as the deformation scale (Stammer 1997), which is suggestive of some form of inverse cascade; indeed, such a cascade has been observed by Scott and Wang (2005) and inferred by Ollitrault et al. (2005). In practice, it may be hard to differentiate marginal supercriticality theories from ar-

guments based on quasigeostrophic turbulence because of the high sensitivity of the eddy statistics, and in particular the eddy diffusivity, to the mean shear, as noted above. Thus, a small increase in the mean shear will lead to a large increase in eddy magnitude and scale, potentially having a large feedback on the mean shear and so constraining its magnitude. Noting this, Zurita-Gotor (2008) studied the sensitivity of the isentropic slope on the forcing in a primitive equation model. His results seemed to suggest that the eddy statistics and the criticality varied continuously with parameters, with no adjustment-like behavior, but the results may have been sensitive to a rather unphysical forcing method. Also, as mentioned above, small-scale convection may also play a role in determining the stratification and hence the criticality, even in the midlatitudes (Jukes 2000).

In this paper we investigate a subset of these issues, in particular the similarities and differences between primitive equation and quasigeostrophic models and whether and under what circumstances we can obtain an inverse cascade in a model that determines its own stratification. Because much of the extant theory, certainly for quasigeostrophic models, is based on a two-level formulation, we restrict our attention to two levels and perform the simulations on a beta plane rather than a sphere, allowing us to change  $f_0$  and  $\beta$  independently, something that cannot be done on the sphere without changing its radius. In the next section we briefly discuss the phenomenology; the numerical model is described in section 3, and sections 4 and 5 evaluate various aspects of the quasigeostrophic theory. Section 6 addresses the vertical eddy heat fluxes in our model and their relationship to the isentropic slope. Finally, section 7 provides a summary and some discussion of the implications and limitations of the present study.

## 2. Phenomenology and notation

It is useful, given the phenomenology sketched in Fig. 1, to briefly summarize the results of the quasigeostrophic turbulence scaling theory, if only to fix notation. The criticality of a uniform quasigeostrophic two-layer flow may be defined by

$$\xi_{\text{qg}} \equiv \frac{U}{\beta \lambda_d^2} = \frac{U k_d^2}{\beta}, \quad (2.1)$$

where  $U = U_1 - U_2$  is the baroclinic wind,  $\beta = \partial f / \partial y$ , and  $\lambda_d$  is the deformation radius based on the layer depth  $H/2$  ( $H$  is the full tropospheric depth), with  $k_d$  being its inverse, the deformation wavenumber. Linear stability analysis of an inviscid uniform shear shows that the flow is baroclinically unstable only when  $\xi_{\text{qg}} > 1$ ; this

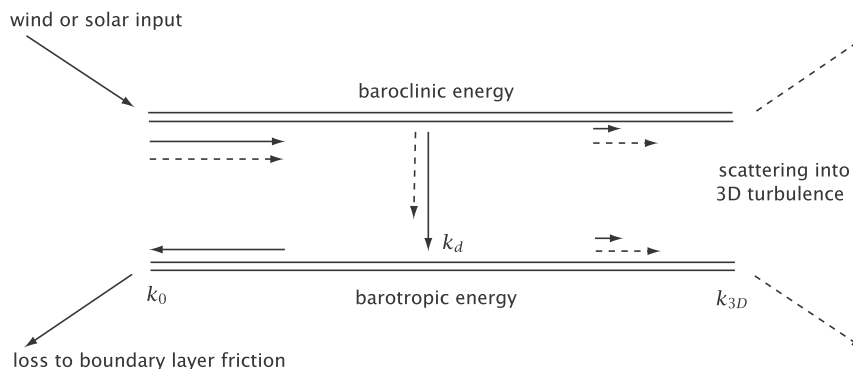


FIG. 1. The phenomenology of two-layer QG turbulence. The horizontal axis represents horizontal wavenumber, increasing to the right, with the two levels representing barotropic (lower) and baroclinic (upper) flow. The arrows represent the passage of energy and enstrophy through the system, with the transfer of energy to the barotropic mode at wavenumber  $k_d$  (the inverse of the radius of deformation). [Following Salmon (1978)].

is a necessary condition for instability if the basic state is steady, and also one that holds for finite amplitude perturbations (Vallis 2006).

The characteristic scale of the most unstable mode for this problem is the deformation scale. However, the scale of the nonlinear eddies may be larger if the flow produces an inverse cascade. Held and Larichev (1996) derived explicit quantitative predictions for various eddy scales in the asymptotic limit  $\xi_{\text{qg}} \gg 1$ , a limit in which eddy momentum fluxes are negligible and diffusion of potential vorticity and of buoyancy are equivalent. If there is an inverse cascade that extends back to a wavenumber  $k_0 \sim 1/L_0$ , then the magnitudes of the barotropic and baroclinic velocities at the large scales— $V_\psi$  and  $V_\tau$ , respectively—are found to be

$$V_\psi \sim \frac{k_d U}{k_0} \quad \text{and} \quad V_\tau \sim U. \quad (2.2)$$

Held and Larichev further assume that the inverse cascade is halted by the beta effect at a wavenumber  $k_0 = k_\beta = \sqrt{\beta/V_\psi}$  (Rhines 1975; Vallis and Maltrud 1993), in which case one finds

$$k_0 \sim \frac{\beta}{U k_d}, \quad V_\psi \sim \frac{k_d^2 U^2}{\beta} \sim \xi_{\text{qg}} U, \quad \text{and} \quad V_\tau \sim U. \quad (2.3)$$

Thus, the velocity field becomes increasingly barotropic as the criticality increases; specifically, the ratio  $V_\psi/V_\tau$  is equal to the criticality. The extent of the inverse cascade  $\gamma$  is then given by

$$\gamma \equiv \frac{L_0}{\lambda_d} = \frac{k_d}{k_0} = \frac{k_d}{k_\beta} \sim \frac{U k_d^2}{\beta} \sim \xi_{\text{qg}}; \quad (2.4)$$

that is, the length of any inverse cascade is directly proportional to the criticality. An eddy diffusivity  $\mathcal{D}$ ,

based on the halting scale and the barotropic velocity, then scales as

$$\mathcal{D} = \frac{V_\psi}{k_0} \sim \frac{U^3 k_d^3}{\beta^2} \sim \xi_{\text{qg}}^3 \frac{\beta}{k_d^3}. \quad (2.5)$$

Note that although the criticality parameter in the above expressions is essentially the same as that determining linear stability in the inviscid two-level quasi-geostrophic model, there is no assumption that the baroclinic eddies grow exponentially or have a normal-mode form, not least because the perturbations are always of finite amplitude. In fact, because the model is inherently nonlinear, the precise mechanisms of linear growth do directly enter the model formulation, which can produce both modal and nonmodal growth (e.g., Farrell 1989). The criticality parameter is really a measure of the extent to which the flow is geostrophically turbulent and thus of the length of the inverse cascade and size of the eddy diffusivity.

These predictions, derived in the asymptotic regime  $\xi_{\text{qg}} \gg 1$ , must break down as  $\xi_{\text{qg}}$  approaches unity, a limit in which the flow becomes stable. Additionally, at low criticality the eddy momentum fluxes are non-negligible and one must distinguish between diffusion of potential vorticity and of buoyancy. Assuming that it is the diffusion of lower-layer potential vorticity that is important, Lapeyre and Held (2003) extended the Held and Larichev theory to order-1 criticalities. The main result is a steepening of the diffusivity power law at low criticality, in agreement with previous empirical results by Pavan and Held (1996). However, an important limitation in both Held and Larichev (1996) and Lapeyre and Held (2003) is spatial homogeneity (note that both studies use domain-averaged eddy statistics), in contrast

with the varying meridional structure found in most geophysical flows. In the spirit of Pavan and Held (1996), Zurita-Gotor (2007) evaluated the homogeneous closures when applied *locally* in the inhomogeneous problem. He found reasonable agreement with (2.5), with no sign of steepening at low criticality as might be expected in the theory of Lapeyre and Held (2003). This difference with the doubly periodic results (that also produce jets) was attributed to the usage of a local rather than domain-averaged closure. Zurita-Gotor also showed that the diffusive closure can capture barotropic governor effects (James 1987) when the full barotropic PV gradient, including the local relative vorticity gradient, is used in place of  $\beta$  in the definition of  $\xi$ . Note that this generalized criticality can now be smaller than 1.

At high criticality, there are also problems with the theory. Because the theory assumes that  $\beta$  halts the inverse cascade, it predicts unbounded eddy scales and fluxes as  $\xi_{\text{qg}} \rightarrow \infty$  ( $\beta \rightarrow 0$ ). A modification of the theory assuming that the cascade is halted by a linear drag fails to predict a finite halting scale. Even with nonzero  $\beta$ , frictional effects are necessary to halt an inverse cascade, as noted by Maltrud and Vallis (1991), yet friction does not enter the above formulation. Thompson and Young (2007) have studied the combined sensitivity of the eddy scales on friction and beta, reporting sensitivity to friction for all values of  $\beta$ . We thus expect the diffusivity to flatten if the halting scale is held nearly constant by friction or if it reaches the domain scales, as is indeed observed in the simulations of Zurita-Gotor (2007) at high criticality.

Given all the shortcomings of the theory even in the quasigeostrophic problem, we cannot expect quantitative agreement with the theoretical predictions in a primitive equation model. Nevertheless, the above description might still provide a useful qualitative description of the primitive equation turbulence phenomenology. By this we mean that, if the QG picture is correct, one would expect to see, as the criticality increases (if it can be made to increase), an increasingly long inverse cascade and eddies whose size is larger than the instability scale, an increasingly barotropic velocity field, and an increasing dominance of eddy available potential energy over baroclinic eddy kinetic energy (EKE; see appendix B), all accompanied by a steep increase in the diffusivity.

Finally, we define a criticality for the primitive equation model as the ratio of nondimensional potential temperature differences in the meridional and vertical direction:

$$\xi_{\text{pe}} = -\frac{f_0}{\beta H} \frac{\partial_y \theta}{\partial_z \theta} = \frac{\Delta_h \theta}{\Delta_v \theta}, \quad (2.6)$$

where  $\Delta_h \theta \equiv -(f_0/\beta) \partial_y \theta$  is a characteristic meridional potential temperature difference and  $\Delta_v \theta = H \partial_z \theta$  is a characteristic vertical potential temperature difference. Using the thermal wind constraint, it is easy to show that this criticality differs from the above definition (2.1) by a factor of 2; that is,  $\xi_{\text{pe}} = \xi_{\text{qg}}/2$  when the potential temperature gradients are evaluated at a midtropospheric level as in this study. Expressed in terms of  $\xi_{\text{pe}}$ , the quasigeostrophic stability limit thus occurs at  $\xi_{\text{pe}} = 0.5$ . However, note that it does not necessarily follow that a two-level primitive equation model also becomes unstable at this value of  $\xi_{\text{pe}}$  because of its different formulation and discretization. In fact, because primitive equation models do not exactly conserve potential vorticity (conservation being especially poor with only two levels) they do not have a Charney–Stern condition and the instability limit, if it exists, is unlikely to be encapsulated in terms of  $\xi_{\text{pe}}$  alone. We will use the primitive equation criticality in the remaining of the paper, dropping the pe subscript for simplicity.

### 3. The model and a control integration

A goal of this study is to assess how well quasigeostrophic turbulence theory does in a primitive equation model that determines its own stratification. Because the quasigeostrophic theory was developed for the simplest case of two equal layers of fluid on the beta plane, we have aimed to make our primitive equation model as similar as possible to this standard configuration. The Massachusetts Institute of Technology (MIT) GCM (Marshall et al. 1997; <http://mitgcm.org/>) is a multilevel primitive equation model that lends itself well to that purpose because of its flexible configuration. We use a two-level, hydrostatic, Boussinesq, beta-plane version of that model, with a linear equation of state (A.5), configured in a zonally re-entrant channel with slippery zonal walls. The specific equations solved are described in appendix A, together with some conservation properties of the system. An important limitation is that there is not a conserved potential vorticity variable.

We added to the MIT GCM simple forcing functions similar to those described by Held and Suarez (1994). We use linear Rayleigh damping with time scale  $\tau_F$  in the lowest model level and force the thermal field by Newtonian relaxation with time scale  $\tau$  to a prescribed radiative equilibrium profile of the form

$$\theta_R = \theta_0 + \frac{\delta_Z}{2} \delta_{k2} - \frac{\delta_Y}{2} \tanh\left(\frac{y - L_Y/2}{\sigma}\right), \quad (3.1)$$

where  $k = 1$  (2) for the lower (upper) level and  $\delta_{k2}$  is the Kroenecker delta. The parameter  $\delta_Y$  controls the

baroclinicity of the radiative equilibrium profile, and the parameter  $\delta_Z$  its stratification. We choose  $\theta_0 = 293$  K, but note that only temperature gradients are dynamically relevant for our Boussinesq fluid. No convective scheme or vertical diffusion is needed or used in the model. For negative  $\delta_Z$ , inspection of the vertical velocity field suggests that grid-size convection stabilizes the flow in regions devoid of baroclinic eddies. Horizontal biharmonic diffusion with  $\nu = 10^{16} \text{ m}^4 \text{ s}^{-1}$  is also used in the momentum equations.

The domain size is  $L_X = 30\,000 \text{ km}$ ,  $L_Y = 18\,000 \text{ km}$ , a width that is much broader than the width of the radiative-equilibrium baroclinic jet ( $\sigma = 1200 \text{ km}$ ) but that may be comparable to eddy scales for the most energetic runs. We use a resolution of  $150 \text{ km}$  both in  $x$  and  $y$  and the fluid depth is  $10 \text{ km}$ . For our control integration we choose typical atmospheric midlatitude values:  $f_0 = 10^{-4} \text{ s}^{-1}$ ,  $\beta = 1.6 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ ,  $\tau = 20$  days,  $\delta_Y = 60 \text{ K}$ ,  $\delta_Z = 40 \text{ K}$ ,  $\tau_F = 5$  days, and a thermal coefficient of expansion  $\alpha_\theta = 3.33 \times 10^{-3} \text{ K}^{-1}$  (see appendix A). These parameters are changed in various other simulations as described below. Finally, we also performed a simulation using the control parameters (except for reduced diffusion) but doubled resolution in both  $x$  and  $y$  to test numerical convergence, finding very similar results.

### Control integration

Figure 2 shows some results for the control run. As in the extratropical troposphere, the baroclinic eddies transport heat northward and upward. Dynamical heating (cooling) dominates at high (low) latitudes and in the upper (lower) level, indicating a reduction of the meridional temperature gradient and an enhancement of the stratification. The eddy heat flux has a lower level maximum but the eddy kinetic energy and eddy momentum flux both display maxima in the upper level. All these features are in qualitative agreement with the extratropical tropospheric climate. Quantitatively, the zonal winds and eddy fluxes also have the right order of magnitude. The main difference from observations and more realistic models is that poleward eddy propagation (equatorward momentum fluxes) dominates in this model, which is why the surface winds are observed southward of the radiative-equilibrium jet, which is symmetric about midchannel. This is consistent with results from other models in Cartesian coordinates, in contrast to the dominance of equatorward propagation on the sphere due to the asymmetry introduced by the metric terms (Balasubramanian and Garner 1997). Finally, the lower right panel shows the quasigeostrophic eddy PV fluxes, calculated using a constant reference  $f_0$  and  $N^2$ , diagnosed from the PE simulation as described

below. Although this is not a quasigeostrophic model, the structure of these fluxes resembles qualitatively those from quasigeostrophic simulations, the main difference being the asymmetry introduced by the eddy momentum flux.

Given the mean stratification, and thus the deformation radius  $NH/f_0$  produced by the primitive equation model, we are in a position to see if the primitive equation results can be replicated by a quasigeostrophic model. Figure 3 compares the PE results with those from a two-layer quasigeostrophic simulation (using the model of Zurita-Gotor and Lindzen 2006). For the quasigeostrophic simulation, we took the value of  $f_0$  at the latitude of maximum lower-level wind in the PE model and the average  $N^2$  over the latitudes with westerlies. The rest of the parameters (frictional and diabatic time scales, radiative equilibrium jet, etc.) are as in the PE simulation. There are two main differences between the models. First, the meridional eddy heat flux is too narrow in the quasigeostrophic model, and the dynamical heating is somewhat underestimated. This might be due to the usage of the same stratification everywhere in the QG model, whereas the stratification is weaker in the PE model away from the jet center (because the eddies enhance the stratification relative to radiative equilibrium, the fluid is more stably stratified over the baroclinic zone than on its sides for the PE equilibrium climate). The second major difference is the magnitude of the upper-level eddy kinetic energy and the structure of the eddy momentum flux, as described above. Despite these differences, the agreement is very good given the different formulations.

Because our control parameters are chosen to replicate typical midlatitude values, one expects the geostrophic assumptions to be reasonably satisfied in this run. However, because this may not always be the case, we have also investigated how well the quasigeostrophic model does in other parameter settings. The results are shown in Fig. 4, which compares the equilibrium criticality and eddy heat flux in both models when changing the diabatic time scale  $\tau$  and the derivative of the Coriolis parameter  $\beta$  (the PE results are discussed in detail later in the text). For each PE simulation, we construct a QG simulation using the mean stratification and reference Coriolis parameter of the PE model, diagnosed as before. One can see that the agreement is again quite reasonable, with the exception of the more strongly forced runs. This could result from the failure of the quasigeostrophic scaling; the isentropic slope is steep and the stratification quite nonuniform in that limit. On the other hand, the simulations varying  $\beta$  show a consistent bias between both models, which could arise from differences in the formulation. But despite these shortcomings,

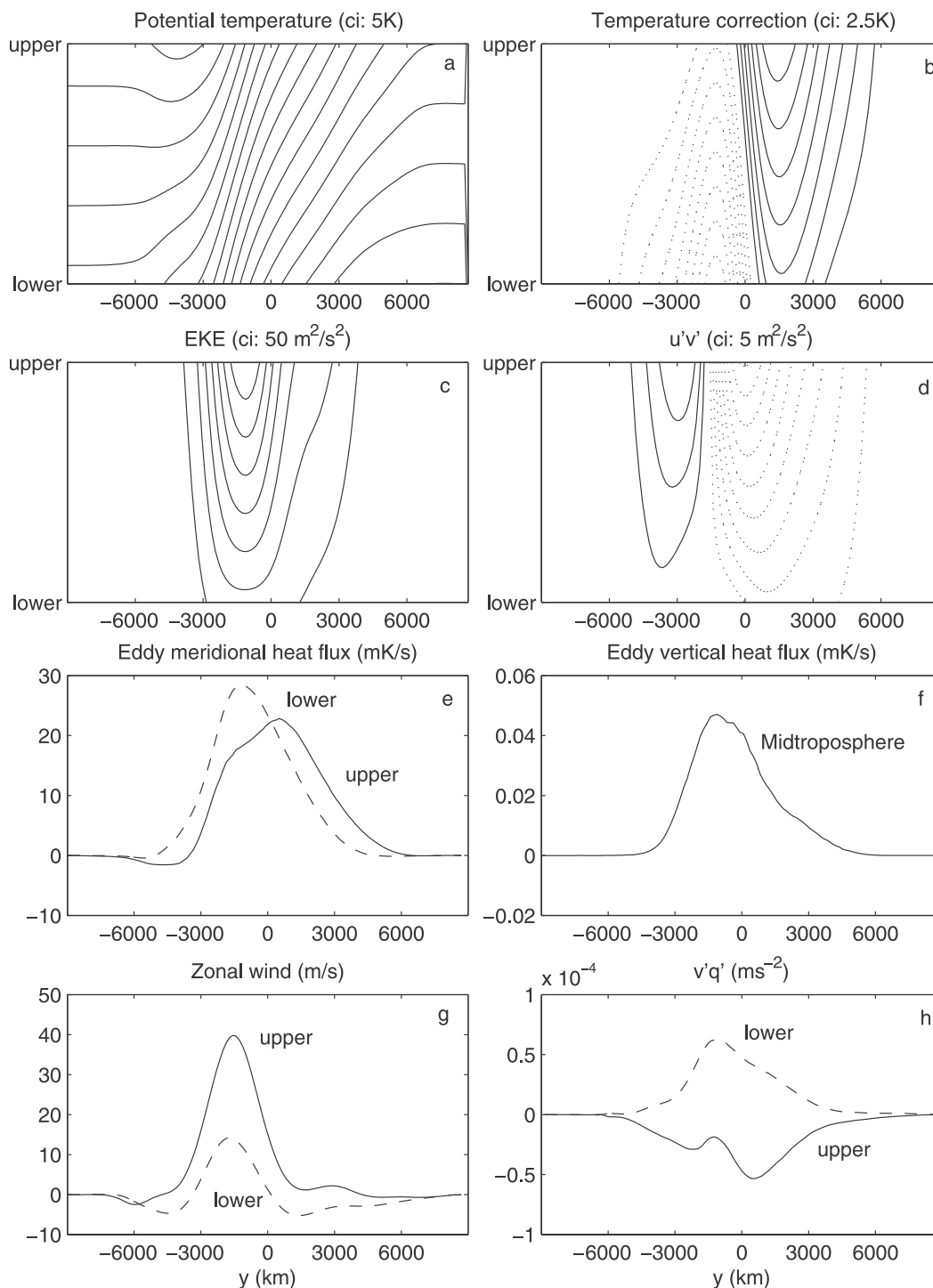


FIG. 2. Description of the control climate: (a) potential temperature; (b) temperature correction from radiative equilibrium; (c) EKE; (d) eddy momentum flux; (e) meridional eddy heat flux; (f) vertical eddy heat flux; (g) zonal-mean wind; and (h) quasigeostrophic eddy PV flux. Note that the contoured fields in (a)–(d) are constructed from the output at the two vertical levels indicated only. The contour interval is indicated in the title and negative contours are dashed.

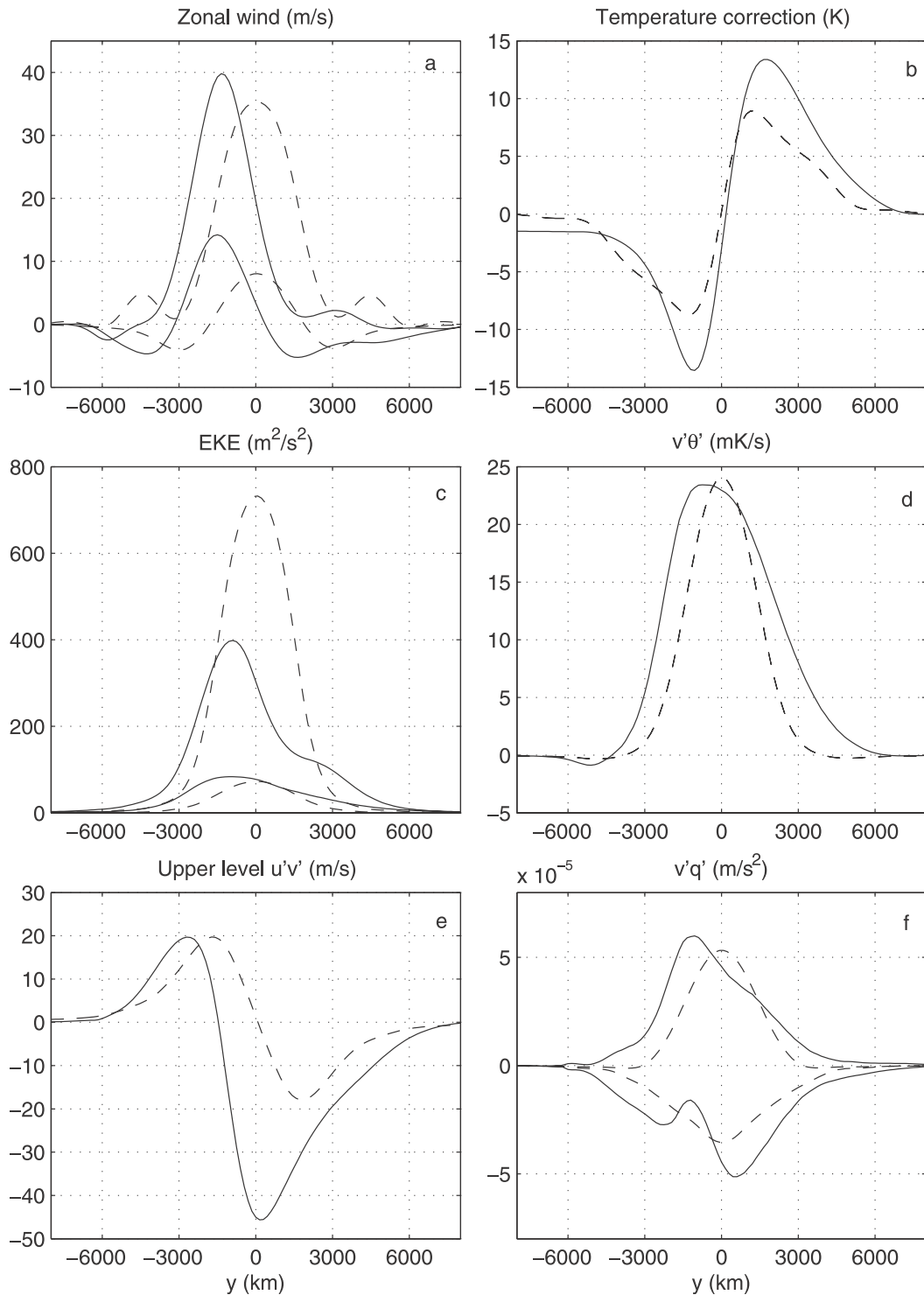


FIG. 3. Comparison of PE and QG simulations using the control setting: (a) zonal wind in both levels; (b) vertically integrated temperature correction from radiative equilibrium; (c) eddy kinetic energy in both levels; (d) vertically integrated eddy heat flux; (e) upper-level eddy momentum flux; and (f) eddy QG potential vorticity flux in both levels. In all panels the solid curves correspond to the PE run and the dashed ones to the QG simulation.

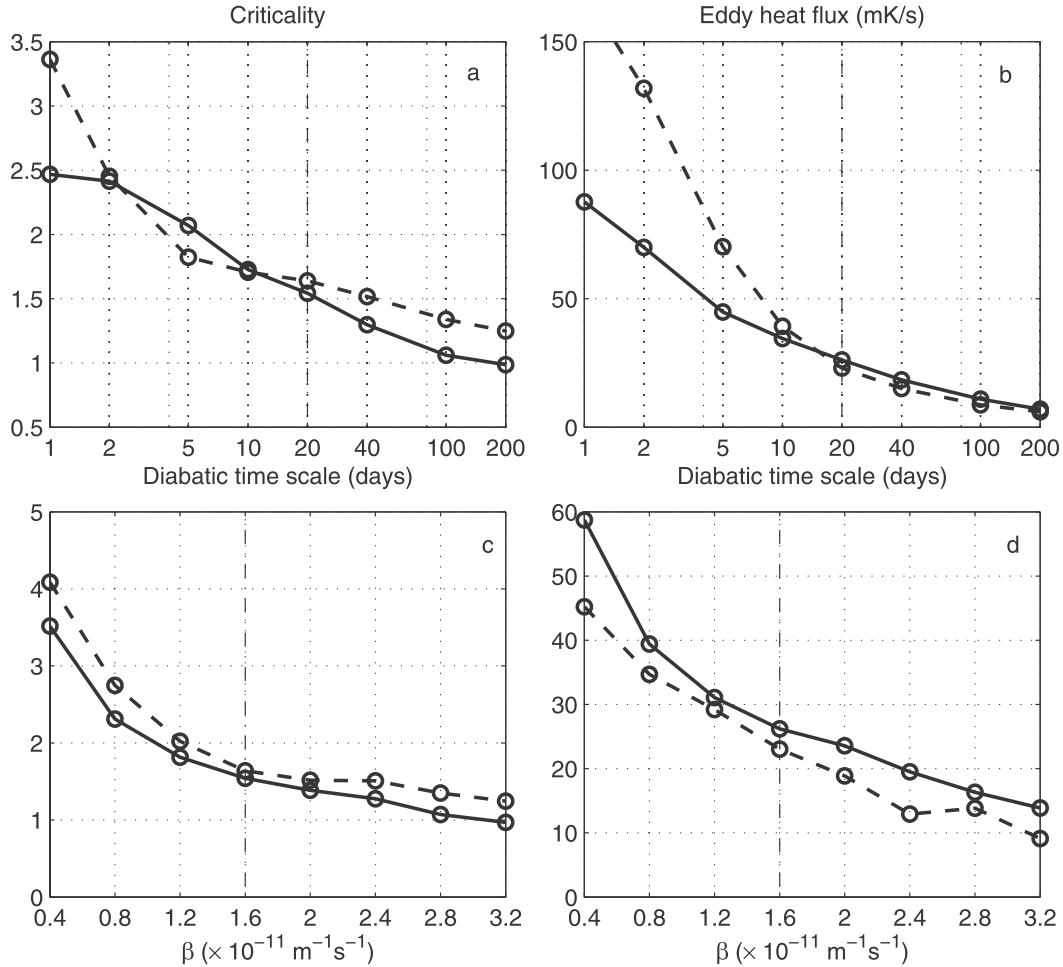


FIG. 4. Sensitivity analysis of the PE–QG comparison: (a) equilibrium criticality and (b) maximum eddy heat flux when the diabatic time scale is changed; (c), (d) as in (a) and (b), but when changing  $\beta$ . The solid curves show results from the PE simulations and the dashed curves results from the QG simulations. The vertical dashed–dotted line indicates the control run, about which both sets pivot.

Figs. 3 and 4 suggest that quasigeostrophic theory can do a good job, *given the stratification*.

#### 4. Effect of parameter variations

To explore further the behavior of the system, we have integrated the system to equilibrium under a large variety of conditions, corresponding to changes in parameter values of three general types, as described below (control values are typed in boldface).

- 1) Changes in the forcing. We changed the baroclinicity ( $\delta_Y = 30, \mathbf{60}, 90, 120, 180, 240$  K) and stratification ( $\delta_Z = -40, -20, 0, 20, \mathbf{40}, 60, 80$  K) of the radiative-equilibrium profile in a 2D parameter sweep. Additionally, we changed the diabatic time scale ( $\tau = 0.5, 1, 2, 5, 10, \mathbf{20}, 40, 100, 200$  days) for the following

( $\delta_Y, \delta_Z$ ) pairs: (**60**, **40**), (90, 40), (120, 40), (180, 40), (60, 0).

- 2) Changes in the rotation parameters/planet size. A convenient feature of this model is that  $f_0$  and  $\beta$  can be changed independently without changing the domain size, which is not so straightforward in spherical geometry. Increasing the  $f_0/\beta$  ratio would be equivalent to simulating the dynamics of a wider planet without actually having to model the entire planet. We performed a 2D parameter sweep of  $f_0$  ( $f_0 = 0.5, 0.75, \mathbf{1}, 1.25, 1.5, 1.75, 2 \times 10^{-4} \text{ s}^{-1}$ ) and  $\beta$  ( $\beta = 0, 0.4, 0.8, 1.2, \mathbf{1.6}, 2, 2.4, 2.8, 3.2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ ). Note that the values of  $f_0$  quoted above refer to midchannel: the actual  $f$  varies with latitude according to  $f = f_0 + \beta y$ .
- 3) Sensitivity to friction. Given that any inverse cascade can only be ultimately halted by friction,

whether or not  $\beta$  is nonzero, we also changed the lower-level frictional damping ( $\tau_F = 0.5, 1.25, 2.5, 5, 10, 20, 40$  days) for five different settings of  $\beta$  ( $\beta = 0, 0.8, 1.6, 2.4, 3.2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ ).

Note that changes in the frictional and diabatic time scale apply to both the eddies and the zonal mean in this work, so that changes in these time scales may have an effect on the eddy damping. As an example, Fig. 5 shows the sensitivity of the model's eddy available potential energy on the diabatic time scale for the values of  $\delta_Y$  indicated. There is a qualitative difference in the sensitivity for short time scales, presumably because of the impact of the eddy damping. This is in contrast with the assumption of inviscid eddy dynamics in the quasigeostrophic theory of Held and Larichev; thus, departure from the theoretical scalings may be expected at high criticality if the forcing is too strong.

Finally, to estimate  $\Delta_h\theta$ ,  $\Delta_v\theta$  in the various integrations, we evaluate  $f_0$  at the latitude of maximum lower-level westerlies and take  $H = 10$  km (the tropospheric depth). The potential temperature gradients  $\partial_y\theta$ ,  $\partial_z\theta$  are calculated at a midtropospheric level (or, equivalently, vertically averaged) and averaged over the latitudes where the lower-level wind is within 10% of its maximum. Although previous studies (e.g., Schneider and Walker 2006) have advocated the use of lower-troposphere gradients instead of midtropospheric ones, we prefer to use the latter because only these are constrained by balance in our model (see appendix A). Nevertheless, the results are qualitatively similar when using lower-level potential temperature gradients in our criticality estimates.

### Equilibrium states

In this section we describe the nature of the equilibrated states of the various integrations performed. For each parameter setting the simulations were spun up to statistical equilibrium for 250 days and then integrated for an additional 550 days, over which period statistics were calculated using bidaily data. In the plots below—for example, Fig. 6—each dot or small circle corresponds to a separate, equilibrated, model integration.

#### 1) EFFECT OF CHANGES IN FORCING

As in a number of previous studies (e.g., Zhou and Stone 1993b; Schneider 2004), changes in the stratification resulting from the vertical eddy heat transport represent a first-order effect for this system; for the same equilibrium meridional temperature gradient, the isentropic slope and criticality change much less than they would if the stratification were prescribed. Figure 6a

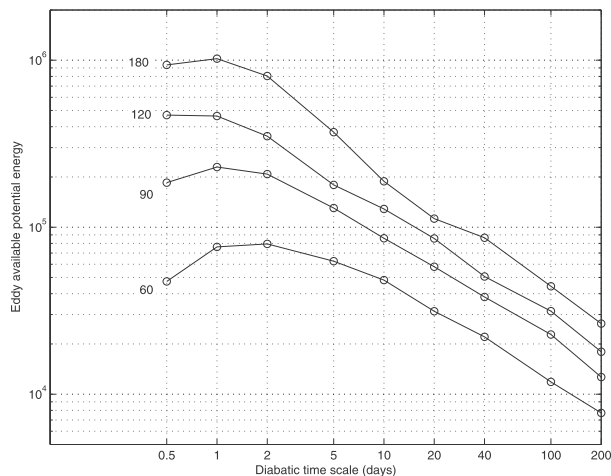


FIG. 5. Eddy available potential energy vs diabatic time scale for sets of simulations with  $\delta_Y = 60, 90, 120$ , and  $180$  K (as indicated) and  $\delta_Z = 40$  K.

shows that the  $(\Delta_h\theta, \Delta_v\theta)$  points are nearly aligned with slope 1, implying that  $\Delta_h$  and  $\Delta_v$  change proportionally and that the criticality stays nearly constant despite the order one changes in both the meridional and the vertical temperature gradients. If the flow were exactly critical, the experimental data points would lie along the dashed-dotted line. Our flow appears to be slightly subcritical ( $\xi \approx 0.7$ ) based on definition (2.6), but  $\xi_{\text{qg}} = 2\xi_{\text{pc}} \approx 1.4$  still exceeds 1.

In general, there is a tendency for changes in the stratification to compensate for the changes in the meridional temperature gradient in our model when  $\delta_Y$  is varied and thus to weaken the sensitivity of the isentropic slope on the forcing. However, the degree of compensation is sensitive to both the structure and the strength of the forcing. Figure 6b shows how  $\Delta_v\theta$  changes with  $\Delta_h\theta$  when both  $\delta_Y$  and  $\delta_Z$  are varied. The solid lines join simulations in which  $\delta_Y$  is varied keeping  $\delta_Z$  constant (the thick line corresponds to the control value  $\delta_Z = 40$  K already shown in Fig. 6a). The dashed lines join simulations with constant  $\delta_Y$  and varying  $\delta_Z$ , as indicated. Note that the time-mean state is always statically stable in the equilibrated state for our simulations (i.e.,  $\Delta_v\theta > 0$ ), even though the model does not have a convective scheme and some of the radiative equilibrium profiles to which we relax are convectively unstable. Figure 6b shows that the criticality  $\xi$  is sensitive in general to the radiative-equilibrium stratification  $\delta_Z$ ; thus, only for very large  $\delta_Y$  do the simulations tend to collapse along a constant criticality line.

It is interesting that departures from constant criticality are found with criticalities both larger and smaller than control. For instance, consider the simulations with weaker baroclinicity ( $\delta_Y = 30$  K, leftmost point for each

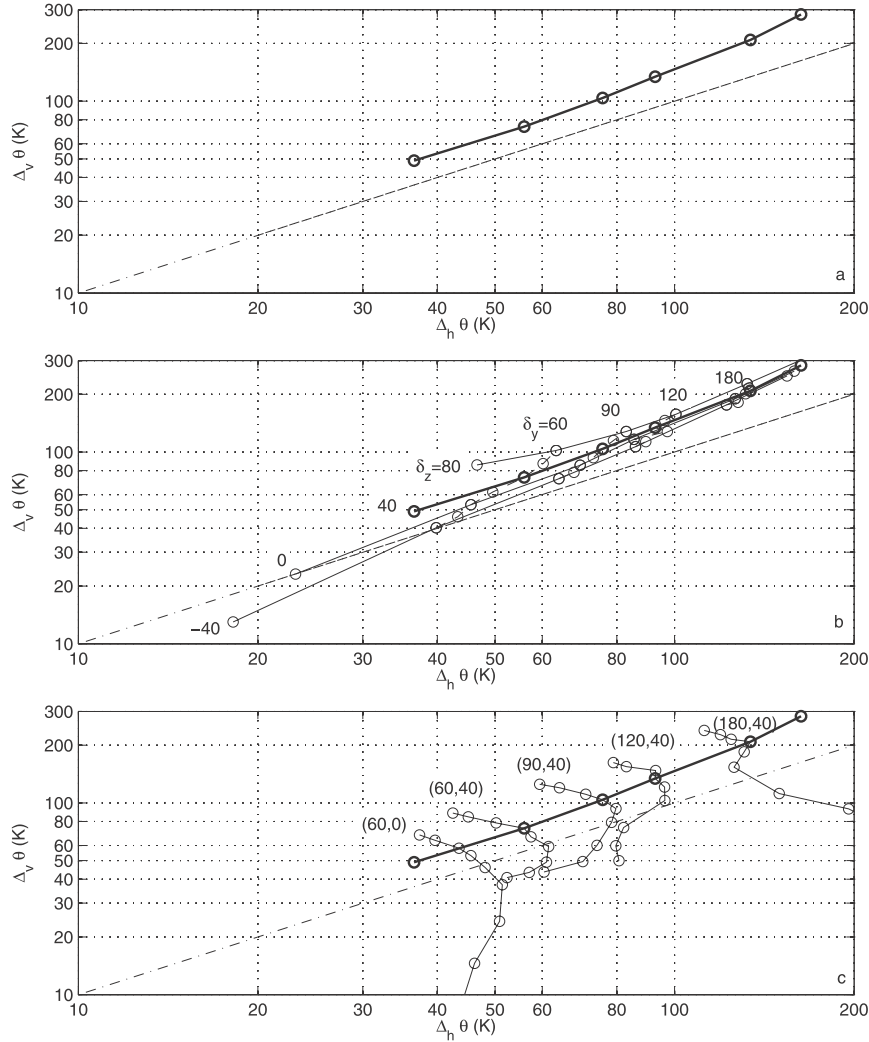


FIG. 6. (a) Vertical vs horizontal temperature gradients in equilibrium changing  $\delta_Y$  in the control setting ( $\delta_Z = 40$  K). (b) As in (a), but also including other  $\delta_Z$  settings. The solid lines join simulations with constant  $\delta_Z$ , as indicated, and varying  $\delta_Y$ ; the dashed lines have constant  $\delta_Y$  and varying  $\delta_Z$ . (c) As in (a), but varying the diabatic time scale  $\tau$  for different  $(\delta_Y, \delta_Z)$  settings, as indicated.

solid line). When  $\delta_Z$  is large ( $\delta_Z = 80$  K), the stratification is far more stable than required by baroclinic adjustment (constant criticality) and is clearly set by the forcing. This would be consistent with the findings of Schneider and Walker (2006), who showed that in the limit in which convection dominates, the stratification can be stronger than required by baroclinic adjustment, rendering a subcritical mean state. What seems to be different in our model is that the mean state can also be supercritical: for convectively unstable or neutral radiative equilibria, the mean stratification is statically stable but weaker than required by marginal criticality. Nevertheless, the sensitivity of the criticality on the stratification is weak as long as  $\delta_Y$  is not too small (note

that the  $\delta_Z$  perturbations considered are unrealistically large). Finally, Fig. 6c shows that the constant criticality constraint is also violated when the diabatic time scale is changed. Interestingly, the sensitivity of  $\Delta_h\theta$ ,  $\Delta_v\theta$  on  $\tau$  in Fig. 6c is qualitatively similar to that reported by Zurita-Gotor (2008) for the multilevel spherical case, even though both the eddy and mean diabatic time scales are changed here.

The sensitivity of the criticality on the forcing is summarized in Figs. 7a,b, which shows contour plots of  $\log \xi$  as a function of the main parameters. Negative (positive) values of the exponent are indicative of subcritical (supercritical) states; we also show for reference the  $\xi = 1$  line (dashed-dotted) and mark the control setting

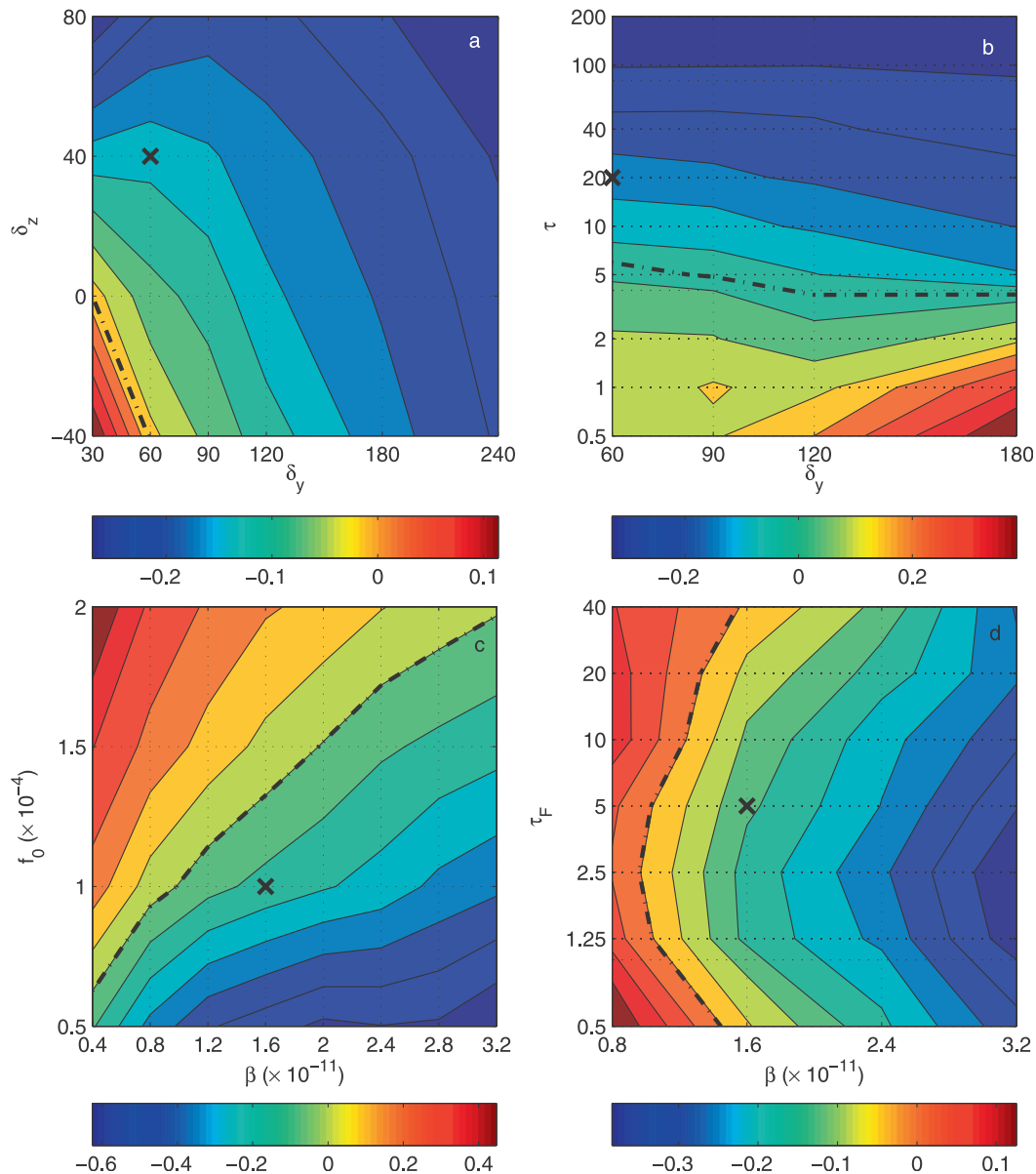


FIG. 7. Logarithm of the equilibrium criticality in the model as a function of the main parameters, shown varying (a)  $\delta_Y, \delta_Z$ , (b)  $\delta_Y, \tau$ , (c)  $\beta, f_0$ , and (d)  $\beta, \tau_F$ . The dashed-dotted line highlights  $\xi = 1$  and the cross marks the control setting. The simulations actually performed are those marked by the intersections of the dotted grid. Any parameter not explicitly varied in a panel is kept at its control setting. Note that the vertical scale is logarithmic in (b) and (d).

with a cross. In general, the sensitivity is weak (hence the use of a logarithmic scale) but also smooth. There is in particular very little sensitivity on the radiative-equilibrium profile, except when  $\delta_Y$  is small (Fig. 7a). A counterintuitive result, also found in Zurita-Gotor (2008), is that the criticality actually decreases with increasing  $\delta_Y$ , which occurs in that study because the gross stability increases faster than the heating with increasing  $\delta_Y$  (this issue is discussed more in section 6). Finally, the uniform sensitivity of  $\log(\xi)$  to  $\log(\tau)$  in Fig. 7b

suggests that this dependency is well approximated by a power law. This is violated for small  $\tau$ , in which case eddy damping effects presumably become important.

## 2) EFFECT OF PLANET SIZE AND THE $F_0/\beta$ RATIO

We have also studied the sensitivity of the criticality on the planet size, which can be achieved by modifying the  $f_0/\beta$  ratio in our beta plane channel, akin to modifying the radius of a spherical planet. In contrast,  $f_0/\beta$  only changes in the spherical case when the baroclinic

zone moves (for a fixed computational domain). This ratio was evidently important in the simulations of O’Gorman and Schneider (2008)—they found that in a broad planet with multiple baroclinic zones the local criticality was larger for the more poleward baroclinic zones. Consistently, we find that the criticality always increases with  $f_0/\beta$  (Fig. 7c)—that is, with the planet radius. Changes in the criticality in this figure arise primarily from changes in  $\Delta_n\theta$  because the stratification is very insensitive to rotation (not shown). Our results are consistent with the notion that  $\beta$  halts the inverse cascade and with the scaling  $L/\lambda \sim \xi$  (Held and Larichev 1996). The inverse cascade broadens both when  $\beta$  decreases (the Rhines scale increases) and when  $f_0$  increases (the Rossby radius decreases), so we expect  $\xi$  to increase in both cases.

### 3) EFFECT OF SURFACE FRICTION

We studied the sensitivity of the criticality on surface friction for different values of  $\beta$ . As noted in the introduction, surface friction is important for dissipating the EKE in the quasigeostrophic turbulence models and must be present for equilibration to occur; moreover, recent results by Thompson and Young (2007) have shown that friction is important in the two-layer quasigeostrophic problem. Figure 7d shows that our model’s criticality is also sensitive to friction and that this dependency is, more surprisingly, nonmonotonic. For fixed  $\beta$  the criticality increases with weakening friction when friction is weak and with strengthening friction when friction is strong. This nonmonotonic behavior is likely due to the different effect of friction on the eddies and on the mean, with the former dominating when friction is strong (Chen et al. 2007). We have also examined (not shown) the behavior of a generalized criticality, constructed by adding the barotropic relative vorticity gradient to  $\beta$  as in Zurita-Gotor (2007). In that case the contours are roughly vertical for  $\tau_F > 2.5$  days, implying weak sensitivity of the generalized criticality in the limit in which mean friction dominates. However, the sensitivity in the strong friction limit remains.

## 5. Length scales and criticality

In this section we examine how the eddy length scales depend on the various parameters and whether and how this relates to the changes in the criticality reported above. In a quasigeostrophic model the criticality can only change through changes in the baroclinic shear. The eddy scale can change if there is an inverse cascade, but the deformation radius is prescribed. In contrast, in our primitive equation model the criticality is affected by changes in the stratification and the deformation

radius is internally determined, which may limit the length of an inverse cascade.

To estimate the deformation radius  $\lambda = NH/f$ , we use the average stratification over the region where the lower-level wind is within 10% of its westerly maximum, the Coriolis parameter  $f_0$  at the location of this maximum, and the full tropospheric depth  $H = 10$  km. The eddy length scale is estimated as  $L = 2\pi/\hat{k}$ , where  $\hat{k}$  is the centroid of the zonal barotropic EKE spectrum (see, e.g., Zurita-Gotor (2007)). This underestimates the eddy scale somewhat because of the weight of the tail. For instance, for an inverse cascade with spectral slope  $n = -3$  halted at wavenumber  $k_0$ , one obtains  $\hat{k} = 2k_0$  (the difference between  $\hat{k}$  and  $k_0$  being larger for shallower spectra). The above conventions produce eddy length scale estimates that are typically 2–5 times larger than the Rossby radii in many of our simulations, which of itself is barely sufficient to make a clear statement that eddy scales are larger than the deformation scale. We shall thus be more concerned here with the sensitivity of the  $L/\lambda$  ratio than with its actual numerical value. We have also performed a few simulations in a larger domain that do produce a clear scale distinction.

We first examine the sensitivity of the length scale on the forcing in Fig. 8. The top panels describe the sensitivity of the length scale, the medium panels that of the Rossby radius, and the bottom ones the logarithm of their ratio. The left panels (Figs. 8a–c) show the sensitivity when  $\delta_Y$  and  $\delta_Z$  are varied. Figure 8a shows that the length scale increases with the radiative-equilibrium baroclinicity  $\delta_Y$  and is very little sensitive to its stratification  $\delta_Z$ . On the other hand, the Rossby radius increases with  $\delta_Z$  but also—and to a larger extent—with  $\delta_Y$  (Fig. 8b). This occurs because the time-mean stratification tends to increase with the time-mean baroclinicity as discussed above. Moreover, a careful inspection reveals that the Rossby radius increases *faster* than the eddy scale with increasing  $\delta_Y$ , so that  $L/\lambda$  decreases (Fig. 8c) and the inverse cascade shrinks despite the increase in the energy level. This is consistent with our above observation that the criticality decreases with increasing  $\delta_Y$ . Figure 8c shows more generally that the dependency of  $L/\lambda$  on  $\delta_Y$  and  $\delta_Z$  has the same structure as that of  $\xi$  (cf. Fig. 7a).

Likewise, Figs. 8d–f show the dependency when  $\delta_Y$  and  $\tau$  are varied. The eddy scale increases with the forcing rate and the radiative-equilibrium baroclinicity, consistent with the notion that the length scale increases with the eddy energy level. This scale saturates at around 11 000 km, which suggests that the eddies might be reaching domain size. On the other hand, the sensitivity of the Rossby radius on  $\tau$  is opposite to that of  $L$ , and  $\lambda$  decreases with increasing forcing rate. This occurs

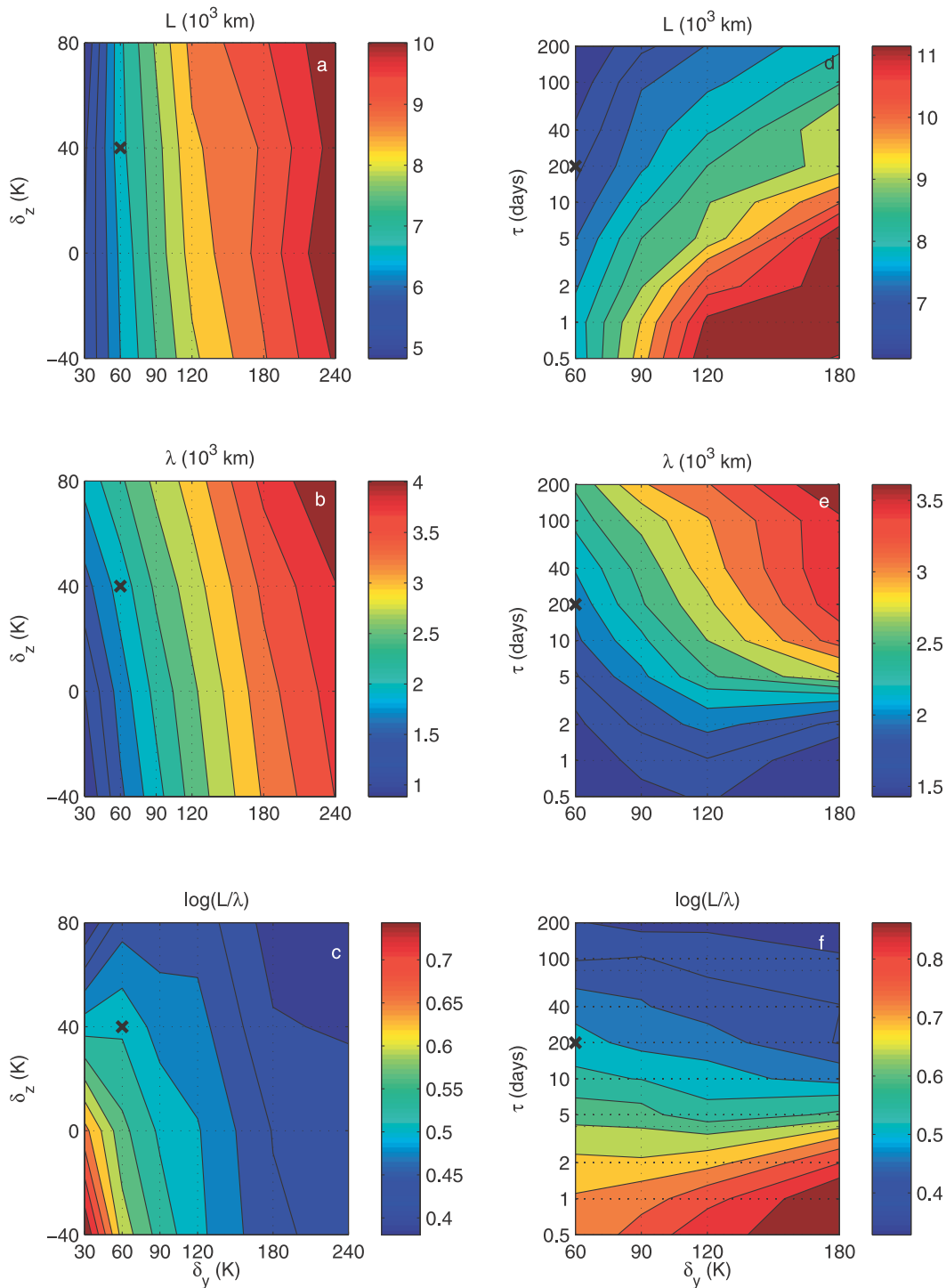


FIG. 8. (a) Eddy scale  $L$ , (b) Rossby radius  $\lambda$ , and (c) their ratio when  $\delta_y$  and  $\delta_z$  are varied. (d)–(f) As in (a)–(c) but when  $\delta_y$  and  $\tau$  are varied. The cross marks the control setting, and any parameter not explicitly varied in a panel is kept at its control value. The simulations actually performed are those marked by the intersections of the dotted grid; the vertical scale in panels (d)–(f) is logarithmic.

because the eddies typically enhance the stratification relative to radiative equilibrium, and departures from this state increase with increasing  $\tau$ . Both facts combined imply that  $L/\lambda$  increases as the forcing rate is accelerated. Figure 8f shows again that this ratio displays sensitivity to  $\delta_Y$  and  $\tau$ , with a structure similar to that of  $\xi$ .

Figure 9 shows the equivalent results when the rotation parameters and friction are changed. We can see that  $\lambda$  is quite sensitive to  $f_0$  and somewhat sensitive to  $\beta$  (Fig. 9b). On the other hand, Fig. 9a shows that the length scale increases with decreasing  $\beta$ , as Rhines' theory would predict. More surprising is the fact that the length scale also increases with decreasing  $f_0$  over the full parameter range, and especially in the small  $\beta$  limit. This is surprising because to the extent that the length scale is defined by the cascade halting scale, the scale of injection would in principle seem irrelevant. What this argument misses is that the equilibrium baroclinicity and the energy level also change (and typically increase) with decreasing  $f_0$ . Nevertheless, note that the length scale increases less than the Rossby radius with decreasing  $f_0$ , so that  $L/\lambda$  still decreases. Figure 9c describes more generally the sensitivity of  $L/\lambda$ , which again compares well to the  $\xi$  sensitivity described in Fig. 7c.

The beta effect is insufficient to stop the inverse cascade by itself. This is especially evident in cases with small beta: it can be seen that the largest eddy scales in Fig. 9a are indeed approaching the width of the domain. Friction must play a particularly important role in this limit [see Vallis (2006, p. 382) for a discussion of the relative importance of friction and beta in arresting the inverse cascade more generally]. Figures 9d–f display the combined sensitivity to  $\beta$  and friction. We can see that the eddy length scale increases with weakening surface friction over the full parameter range, most notably when  $\beta$  is small. On the other hand, the sensitivity of the Rossby radius on friction is nonmonotonic, suggesting again that there might be two different regimes depending on whether eddy or mean friction dominates. The sensitivity of  $\lambda$  to friction is fairly weak (note the narrower  $\lambda$  range compared to other panels). The sensitivity of  $L/\lambda$  is thus dominated by the sensitivity of  $L$ , with  $L/\lambda$  increasing both with weakening friction and with decreasing  $\beta$ . The structure of the  $L/\lambda$  dependency is somewhat similar to that of  $\xi$  (cf. Fig. 7d), except when friction is strong and eddy friction dominates.

The results presented above suggest that the eddies in these runs have a larger scale than the most unstable mode. To confirm this impression, Fig. 10 compares the scale of the most unstable mode of the time-mean state and that of the nonlinear eddies. This is done for two different simulations: the control simulation and a higher-criticality run with  $\delta_Y = 120$  and  $\tau = 1$  (all other

parameters adopt control values). The most unstable modes were calculated using an unforced version of the model in channels of varying length, not necessarily quantized. We found that the mode with fastest growth rate peaks at  $k = 6.25$  in both simulations (Fig. 10, top panels). We also computed the one-point correlation of the upper-level meridional velocity with a midchannel base point to estimate the scale of the eddies (e.g., Chang 1993) in the actual nonlinear simulations. The results (Fig. 10, bottom panels) show that the eddies are already larger than the most unstable mode for the control run and expand further in the high-criticality run.

#### a. Criticality dependencies

The results presented so far are qualitatively consistent with the quasigeostrophic theory. In this subsection we evaluate this agreement more quantitatively, testing observed criticality dependencies in our model against some of the theoretical predictions. Following Zurita-Gotor (2007), we will use a generalized criticality constructed using a generalized beta that also includes the lateral gradient of relative vorticity:

$$\xi^* = -\frac{f_0}{\beta^* H} \frac{\partial_y \theta}{\partial_z \theta} \quad \text{and} \quad \beta^* = \beta - \frac{1}{2} \partial_{yy} (U_1 + U_2). \quad (5.1)$$

As shown by Zurita-Gotor (2007), this redefinition of  $\xi$  can capture changes in the mean state associated with barotropic governor effects. Usage of this generalized criticality reduces the scatter in the figures to follow, in some cases appreciably. However, the results are qualitatively similar when the standard criticality is used, except as described below.

The top panel of Fig. 11 displays  $L/\lambda$  as a function of  $\xi$  for all our runs, with the exception of those in which surface friction is varied. Plotting conventions in this and subsequent similar figures are as follows: We use the same marker to indicate sets in which the same parameter is varied (the marker–parameter correspondence is indicated in the caption). Additionally, sets based on the control parameters always have filled markers. For instance, there are five different sets with circle markers, all describing the sensitivity to  $\tau$ , but each corresponding to a different setting. Of these five sets, only one has filled circles, the one with the control parameters  $\delta_Y = 60$  K and  $\delta_Z = 40$  K. The small criticality values along the  $x$  axis may be surprising at first because two-layer QG models are stable for  $\xi_{\text{qg}} < 1$  and the Larichev and Held theory furthermore assumes  $\xi_{\text{qg}} \gg 1$ . However, note that the criticality plotted here differs from  $\xi_{\text{qg}}$  in two respects: we use the primitive equation definition  $\xi_{\text{pe}} = 0.5\xi_{\text{qg}}$  and also employ a

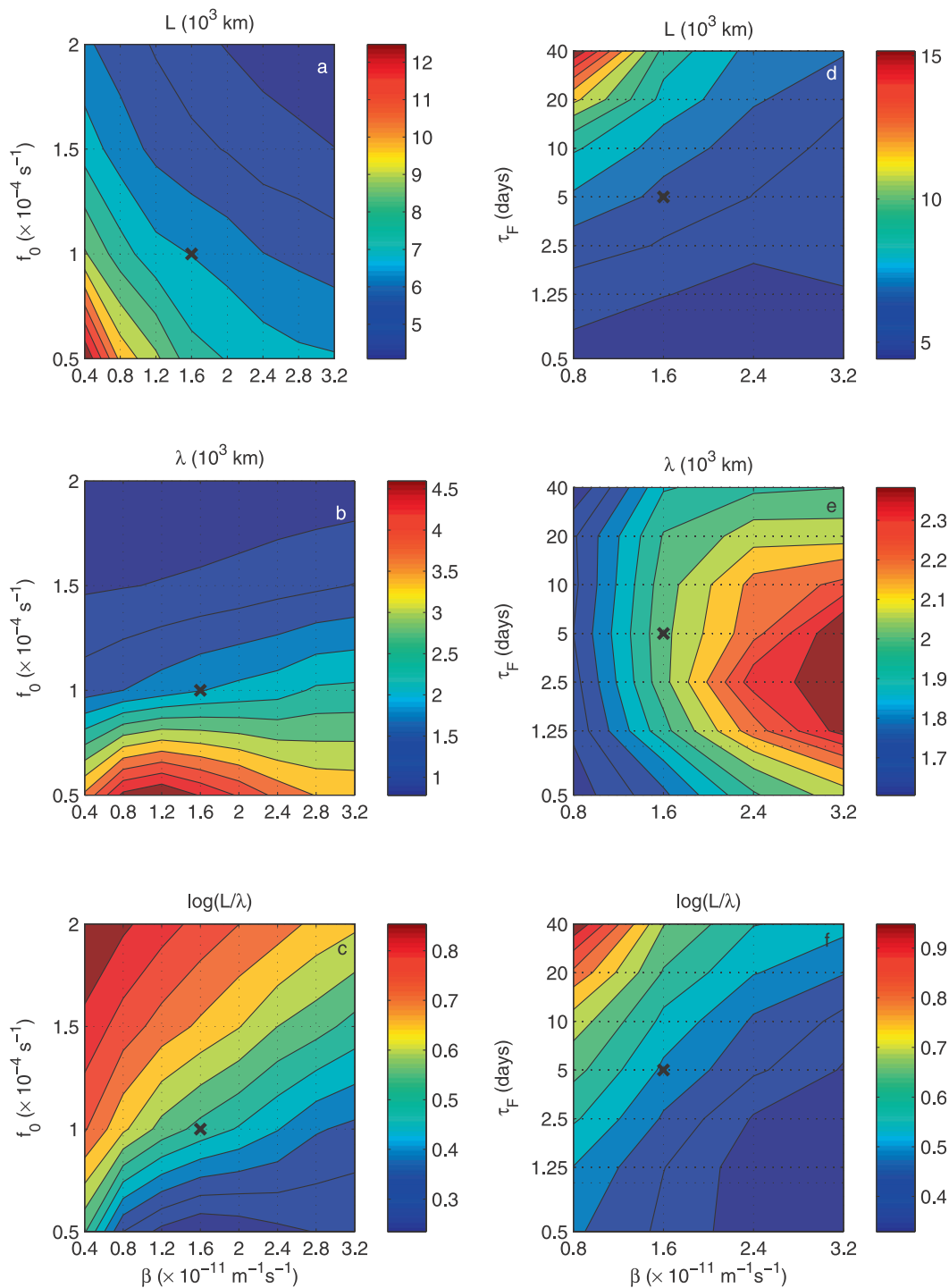


FIG. 9. As in Fig. 8, but when varying (a)–(c)  $\beta$ ,  $f_0$  and (d)–(f)  $\beta$ ,  $\tau_F$ .

generalized  $\beta^*$ . Because  $\beta^* \sim O(2\beta)$  in most of our runs, the QG stability limit is thus roughly associated with  $\xi \approx 0.25$  in terms of the criticality used here. This value is provided for reference only: note that the

primitive equation model has different stability properties (see section 2).

Despite some scatter, Fig. 11 suggests that the  $L/\lambda$  ratio scales with the criticality  $\xi$  in our model, albeit with

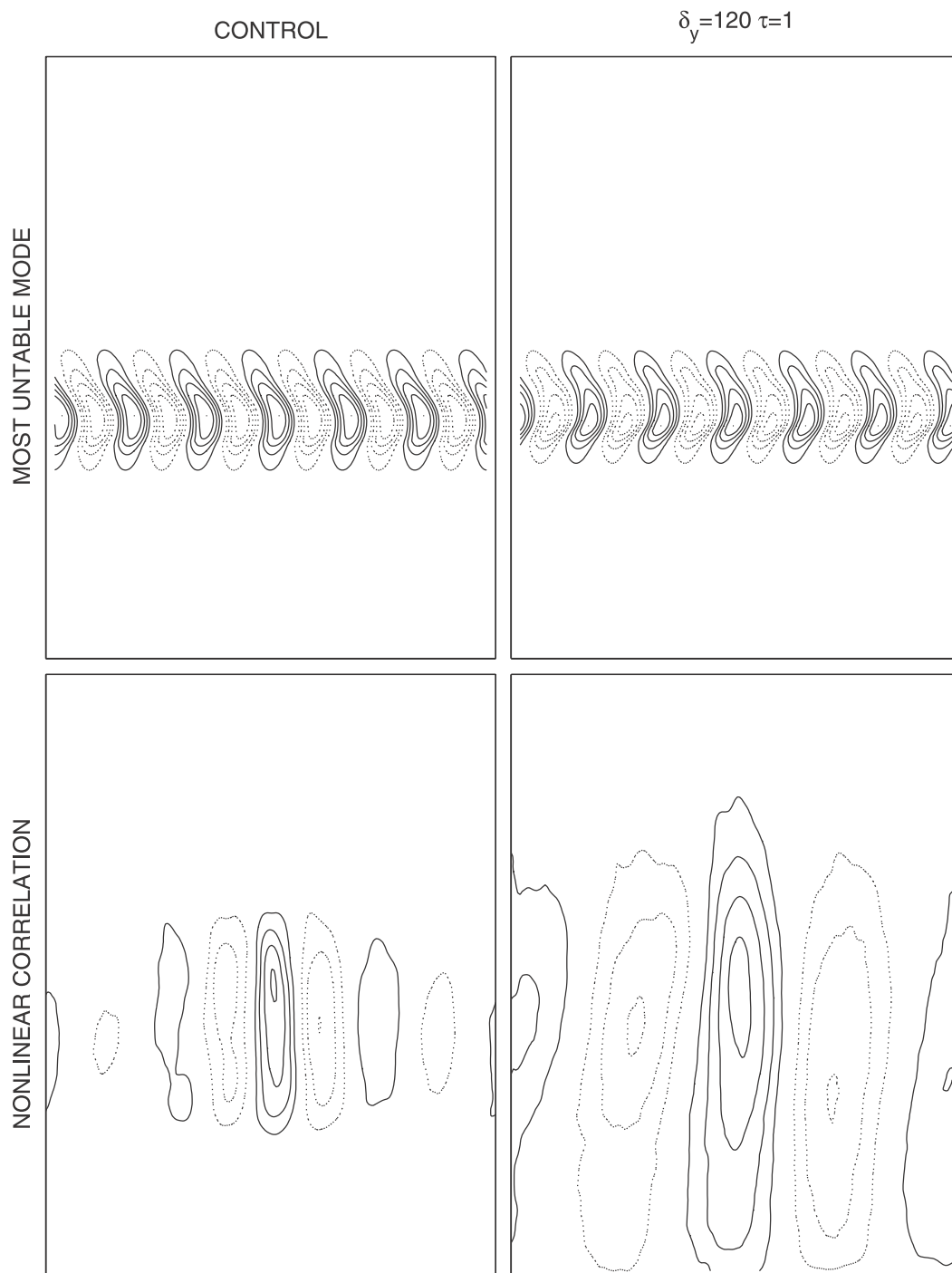


FIG. 10. (top) Upper-level meridional velocity for the most unstable mode and (bottom) the one-point correlation of upper-level meridional velocity in the nonlinear results: (left) the control run and (right) a high-criticality simulation with  $\delta_y = 120$  and  $\tau = 1$ .

a weaker sensitivity than predicted by the theory. For large  $\xi$  the different sets both diverge and flatten, which suggests that the theory is breaking down in that limit. There are several reasons why the theory might fail at

high criticality, as discussed in section 2. For the sets varying  $\tau$  (circles), eddy diabatic damping is likely important in the large  $\xi$  limit. For the sets varying  $\beta$  (triangles), friction and/or the domain size might be more

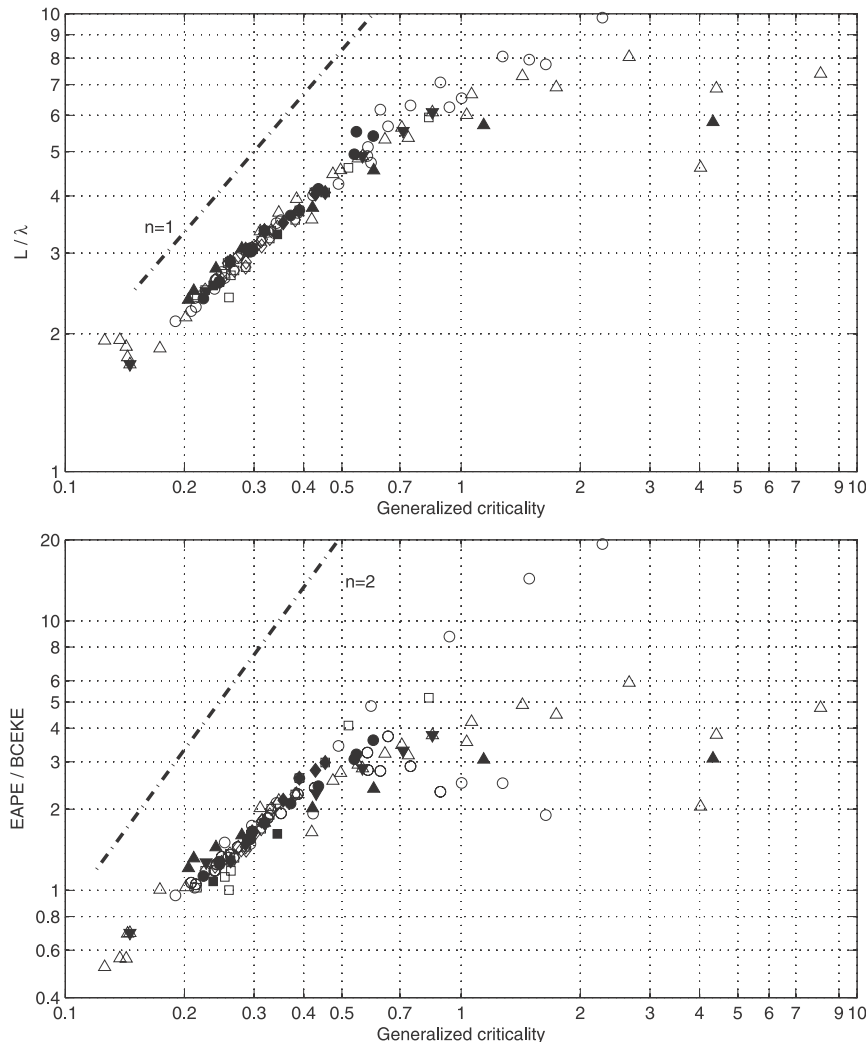


FIG. 11. (top)  $L/\lambda$  and (bottom) the ratio between EAPE and BCEKE for all runs except varying friction. Each marker corresponds to a set of simulations in which a single external parameter is varied:  $\tau$ , circles;  $\delta_Z$ , diamonds;  $\delta_Y$ , squares;  $\beta$ , triangles;  $f_0$ , inverted triangles). In most cases, there are several sets varying the same external parameter but with different base settings. We use filled markers to emphasize the control set (e.g., the set that has control values for all parameters but the one that is varied).

important than the Rhines effect in halting the inverse cascade when  $\beta$  is small. We have not included in this figure the sets of runs varying friction for clarity because the sensitivity in these sets does not adhere to the general behavior. For the QG case, Zurita-Gotor (2007) showed that the effect of friction on the mean could be incorporated to the theory by simply generalizing  $\beta$  to include the relative vorticity gradient. However, this does not seem to be sufficient in our case (note that we are already using this generalized criticality), probably because changes in friction also apply to the eddies in this work.

A different measure of the inverse cascade based on global statistics rather than local averages was proposed

by Schneider and Walker (2006), as the ratio between eddy available potential energy (EAPE) and baroclinic eddy kinetic energy (BCEKE). These authors showed that there was an approximate equipartition between both eddy energies in their multilevel primitive equation simulations, consistent with their constant criticality and with their finding  $L \approx \lambda$ . In some contrast, Zurita-Gotor (2008) found that  $L/\lambda$  increased with  $\xi$  in his model but that the  $EAPE/BCEKE$  ratio displayed an erratic behavior, with no evidence one way or the other for an inverse cascade. [Appendix B discusses the relation between  $L/\lambda$  and  $EAPE/BCEKE$  and provides an explanation for why these two diagnostics might lead

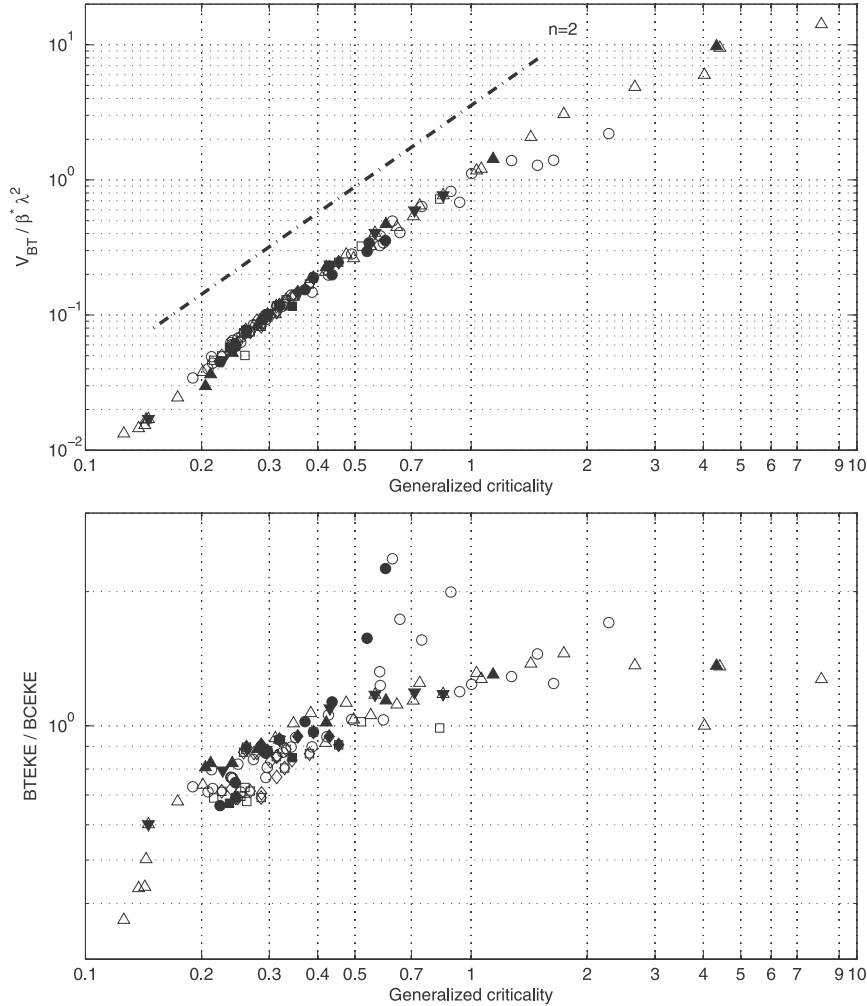


FIG. 12. As in Fig. 11, but for (top) the normalized energy level and (bottom) the ratio between barotropic and baroclinic EKE.

to different conclusions in the work of Zurita-Gotor (2008)]. Figure 11 shows the dependence of the EAPE to BCEKE ratio on the criticality in our model. The details of the EAPE calculation are discussed in appendix A—note that only the component associated with anomalies in the vertically integrated potential temperature [first term in (A.10)] is included in this calculation. The scatter is larger than before, probably because, as global integrals, EAPE and BCEKE also include far-range influences. Nevertheless, Fig. 11 shows that the EAPE/BCEKE ratio increases monotonically with the criticality in our runs, consistent with the broadening of the inverse cascade.

Together with the expansion of the eddies, quasigeostrophic theory predicts an increase in their energy level as the criticality increases. In the simplest case,  $V_{\psi}/U \sim \xi$ , where  $V_{\psi}$  is a characteristic barotropic eddy velocity and

$U$  is the baroclinic shear; equivalently,  $V_{\psi}/(\beta\lambda^2) \sim \xi^2$ . Figure 12 (top) shows good agreement with this prediction in our simulations, using the square root of the barotropic EKE (averaged over the baroclinic zone only) to estimate  $V_{\psi}$ . The quasigeostrophic theory also makes very specific predictions about eddy energy partition. The first prediction—namely, that the EAPE/BCEKE ratio increases with criticality—was already shown to be satisfied in Fig. 11. Additionally, the theory predicts a barotropization of the flow, with the barotropic EKE dominating over the baroclinic EKE at high criticality. However, Fig. 12 (bottom) shows that the barotropization of the flow is moderate in our runs. This suggests that our simulations are still far from the asymptotic limit in the theory, perhaps because of friction.

We have finally examined the dependence of the eddy diffusivity. The geostrophic turbulence picture is likely

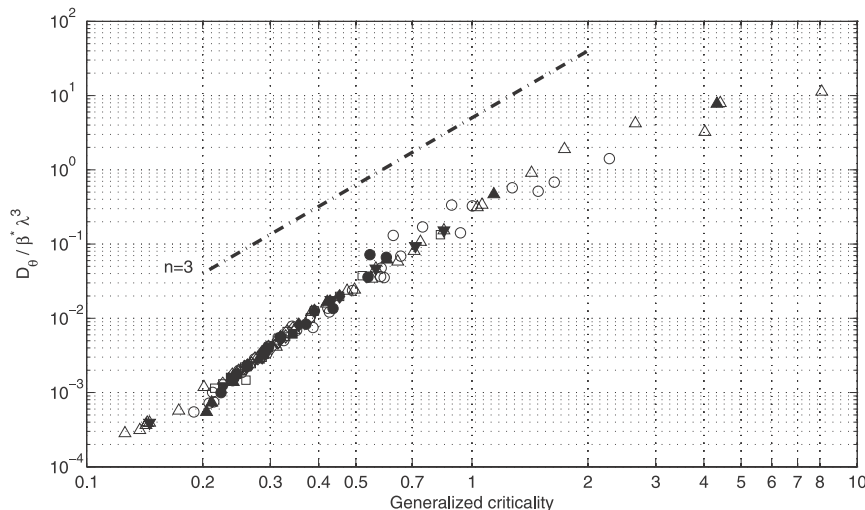


FIG. 13. As in Fig. 11, but for the empirical diffusivity for temperature.

valid only at high criticality, whereas at low criticality the eddy momentum fluxes become important and the diffusivities for upper- and lower-layer PV and for potential temperature are generally different. For the two-layer QG model, Lapeyre and Held (2003) and Zurita-Gotor (2007) propose using the lower-layer PV diffusivity. We have also tried this closure, estimating the quasigeostrophic potential vorticity (QGPV) fluxes using the equivalent two-layer model with the same mean state and eddy fluxes (as in Fig. 2h). However, this does not bode well for our simulations because in some of these simulations (particularly with large  $\tau$ ), the lower-level QGPV gradient becomes positive definite in the equivalent QG model, and the PV fluxes are upgradient (these runs also have  $\xi^* < 0.25$  using the generalized criticality as in Figs. 11–13). Because QGPV is not really conserved in this model, there does not seem to be any advantage in using the lower-layer QGPV over  $\theta$  for the diffusive closure. Figure 13 shows the sensitivity  $\mathcal{D}_\theta / \beta \lambda^3$  for our simulations. We can see that the experimental data points are reasonably aligned, producing a closure similar to the Held and Larichev prediction even though the assumption  $\xi \gg 1$  is not strictly satisfied. It is remarkable that the cubic power law appears to hold for all criticalities, with no obvious steepening at low criticality as in the work of Lapeyre and Held (2003). Equivalent plots using the standard criticality  $\xi$  instead of  $\xi^*$  show some hint of low-criticality steepening but also display more scatter (not shown). The lack of steepening in our closure is consistent with the inhomogeneous QG results of Zurita-Gotor (2007), who attributed the different sensitivity from the doubly periodic simulations of Lapeyre and Held (2003) to the use of local estimates instead of domain-averaged diagnostics.

### b. Spectral slopes and inertial ranges

If in geostrophic turbulence there is a classical, up-scale energy transfer of barotropic EKE, then one might expect to see an energy spectrum with spectral slope of  $n = -5/3$  at scales larger than the deformation radius, characteristic of the inverse energy cascade of 2D turbulence. One might also expect to see a spectral slope of  $n = -3$  at scales smaller than the deformation radius, characteristic of the enstrophy cascade of two-dimensional turbulence.

Neither of these spectra is unambiguously seen in the earth's atmosphere. The  $-5/3$  spectra is not seen at all; rather, at scales larger than the deformation radius the spectrum is rather flat and uneven, with no distinct slope. At scales smaller than the deformation radius the spectrum does follow an approximate  $-3$  law (e.g., Gage and Nastrom 1986), but its extent is too small for this to be definitively ascribed to an enstrophy cascade. There is, however, some evidence for an inverse cascade and a  $-5/3$  spectrum in the ocean (Ollitrault et al. 2005; Scott and Wang 2005), consistent with the observation that oceanic mesoscale eddies are often observed to be much larger than the local deformation radius, but there are insufficient observations to detect a possible enstrophy cascade at small scales.

Why is there no  $-5/3$  spectrum in the atmosphere? The proximate reason is that the deformation radius is too large, being of the same order of magnitude as the radius of the earth itself, so there is no room for an inverse cascade to develop. However, this is an unsatisfactory explanation because it begs the question as to why the deformation radius has the scale that it does. If the atmosphere does evolve toward states of marginal

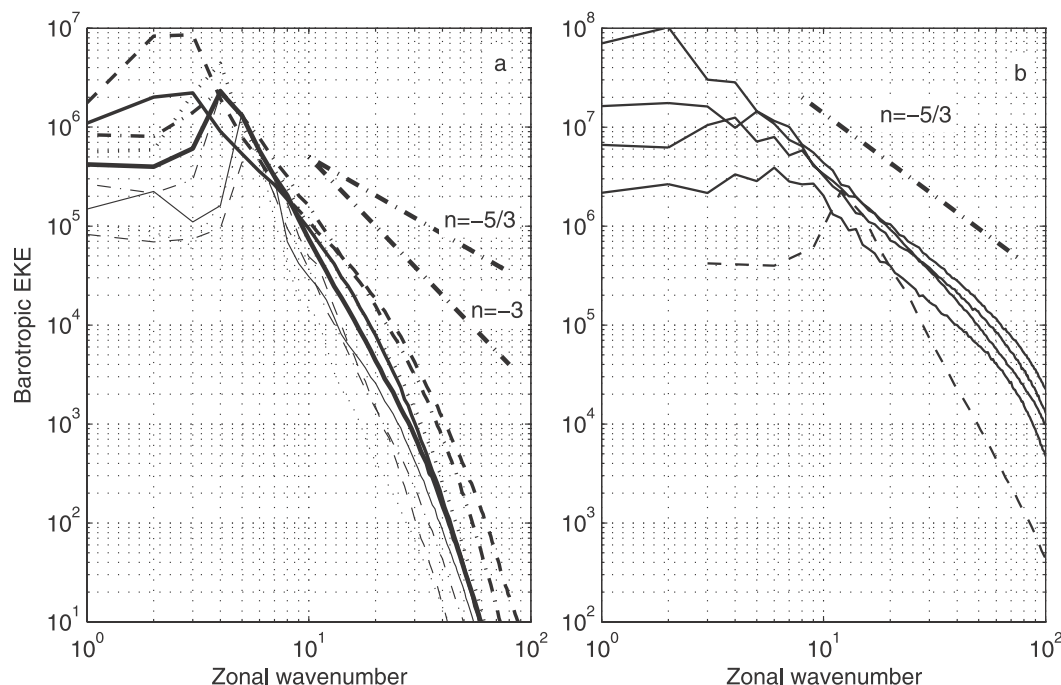


FIG. 14. (a) Zonal barotropic EKE spectra for the control run (solid thick) and for runs with extreme values of  $\beta$  ( $\beta = 0.4, 3.2$ ; solid thin);  $\delta_Y$  ( $\delta_Y = 30, 240$ ; dashed);  $\delta_Z$  ( $\delta_Z = -40, 80$ ; dashed-dotted); and  $\tau$  ( $\tau = 5, 200$ ; dotted). For each couplet, the most energetic run is shown with a thicker line. (b) Zonal barotropic EKE spectra for the four simulations with enhanced scale separation described in the text (solid), with the control spectrum in the standard channel shown (dashed) for reference.

supercriticality, and if any inverse energy cascade is halted by the beta effect, then—as emphasized by Schneider and Walker (2006)—no inverse cascade will arise, even if the geometry does not forbid it. But even in the presence of well-developed geostrophic turbulence with a deformation radius at some intermediate scale, such classical spectral slopes may not arise because of all the other conditions needed, in particular the need for spectrally localized energy and enstrophy injection and for dissipation locations spectrally far removed from the injection.

For these reasons, it is useful to explore the energy spectra in our simulations; Fig. 14a shows the barotropic EKE spectra in our model for a few representative runs (see caption for details). The spectral slope is typically  $n = -3$  or steeper, far from the theoretical  $n = -5/3$ . This is in spite of the hint of an inverse cascade or at least a transfer of energy to larger scales, as noted in previous sections.

Although differences with the theory might be due to non-QG feedbacks, it seems more likely that the model (like the earth's atmosphere) lacks enough scale separation to produce an inertial range, and the domain size is affecting the spectrum. To examine this, we conducted another set of integrations aimed at producing a more

vigorous inverse cascade and increasing the scale separation between  $\lambda$  and  $L$  and between the latter and the domain scale. We use a channel 3 times as wide and as large, keeping the same grid size of 150 km. We take  $\delta_Y = 90$  K,  $\delta_Z = 0$ ,  $f_0 = 3 \times 10^{-4} \text{ s}^{-1}$ ,  $\beta = 12 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ , and  $\tau_F = 15$  days, keeping  $\sigma$  and  $\alpha_\theta$  unchanged. We consider four different values of the forcing time scale:  $\tau = 1, 5, 20$ , and 80 days. The main difference from the standard setting is the expanded domain, the reduced friction, and the large value of  $f_0/\beta$  (corresponding to a planet 4 times as large), more as a result of the large increase in  $f_0$  than of the small reduction in  $\beta$ . We chose not to reduce  $\beta$  further to try to ensure that the eddies do not reach the domain scale. On the other hand,  $f_0$  cannot be increased further with our 150-km resolution if we want to resolve the Rossby radius. Table 1 shows the main diagnostics for these four simulations. The criticality is much larger than reached with the standard channel, and the eddy scale and Rossby radius are also better separated. The barotropization of the flow is now more obvious (even the eddy momentum flux is nearly barotropic), and the diffusivity increases by an additional factor of 30 compared to the largest values before. Interestingly, Fig. 14b shows much shallower spectra in these runs, approaching the theoretical  $-5/3$  prediction.

TABLE 1. Diagnostics for the runs performed for the wide channel;  $\beta^*$  is a generalized  $\beta$  that also includes the barotropic relative vorticity gradient, and  $\xi^*$  is the criticality based on this beta. Other symbols and acronyms have standard meanings. Here, barotropic EKE is abbreviated BTEKE.

| $\tau$ | $\xi$ | $\xi^*$ | $L$ (km) | $\lambda$ (km) | $\frac{L}{\lambda}$ | $\frac{\text{EAPE}}{\text{BCEKE}}$ | $\frac{\text{BTEKE}}{\text{BCEKE}}$ | $\frac{V_\theta}{\beta^* \lambda^2}$ | $\frac{\mathcal{D}}{\beta^* \lambda^2}$ | $\frac{\max(\overline{u'v'_z})}{\max(\overline{u'v'_1})}$ |
|--------|-------|---------|----------|----------------|---------------------|------------------------------------|-------------------------------------|--------------------------------------|-----------------------------------------|-----------------------------------------------------------|
| 1      | 33.4  | 21      | 18 900   | 157            | 121                 | 335                                | 19.2                                | 90.7                                 | 331                                     | 1.00                                                      |
| 5      | 9.3   | 5.6     | 9400     | 385            | 24                  | 48                                 | 5.5                                 | 9.6                                  | 7.4                                     | 0.83                                                      |
| 20     | 3.9   | 2.9     | 8600     | 513            | 17                  | 28                                 | 4.2                                 | 5.6                                  | 3                                       | 0.71                                                      |
| 80     | 2.4   | 1.8     | 7700     | 590            | 13                  | 19                                 | 3.2                                 | 2.7                                  | 0.9                                     | 0.64                                                      |

## 6. Vertical heat flux closure

The preceding sections have shown that quasigeostrophic theory works reasonably well when the appropriate stratification is used. To have a fully closed theory, however, one also needs to predict the stratification, or at least the eddy vertical heat fluxes. The stratification would then be determined by a balance between these fluxes and the external diabatic heating (Stone 1972; Held 2007).

In principle, QG theory only provides information on the eddy meridional heat flux  $\overline{v'\theta'}$ . However, one can also calculate the eddy vertical heat flux if one assumes that the eddies are adiabatic because then the net flux is aligned along the isentropic slope. This can be seen by constructing an equation for  $\overline{\theta'^2/2}$ :

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\overline{\theta'^2}}{2} \right) + \overline{v'\theta'} \overline{\theta_y} + \overline{w'\theta'} \overline{\theta_z} + \frac{\partial}{\partial y} \left( \overline{v \frac{\theta'^2}{2}} \right) \\ + \frac{\partial}{\partial z} \left( \overline{w \frac{\theta'^2}{2}} \right) = \overline{Q'\theta'}, \end{aligned} \quad (6.1)$$

where the fourth and fifth terms on the left-hand side include the advection from both eddies and mean:  $v = \bar{v} + v'$ ,  $w = \bar{w} + w'$ . If one further assumes that (i) the eddies are adiabatic so that the right-hand side is zero and (ii) the creation of  $\overline{\theta'^2/2}$  is balanced *locally* by its destruction, then one obtains in the time mean

$$\frac{\overline{w'\theta'}}{\overline{v'\theta'}} \sim -\frac{\overline{\theta_y}}{\overline{\theta_z}}, \quad (6.2)$$

that is, the total eddy heat flux is aligned along the isentropic slope. We emphasize that our rationale for neglecting the last two terms on the left-hand side of (6.1) is locality, a standard assumption in diffusive closures, not the smallness of eddy amplitude.

Assuming adiabaticity is not the only way to obtain a relationship between eddy fluxes and isentropic slope—in linear baroclinic instability, the heat flux may be directed at half the angle of the isentropic slope [e.g., Green (1970); more discussion of the general matter is

given in Vallis (2006, chapter 10)]. But, in any case, having a proportionality relationship between vertical flux and isentropic slope allows us to calculate  $\overline{w'\theta'}$  from the diffusive closure for the meridional heat flux,

$$\overline{w'\theta'} \sim -\overline{v'\theta'} \frac{\overline{\theta_y}}{\overline{\theta_z}} \sim \beta \lambda^3 \xi^3 \frac{\overline{\theta_y}^2}{\overline{\theta_z}}, \quad (6.3)$$

by using the diffusive closure for  $\mathcal{D}_\theta$ , namely (2.5), with  $\lambda = k_d^{-1}$ . After some rearrangement this may be written as

$$\overline{w'\theta'} \sim \xi^5 \left( \frac{\Theta_0 \beta^3 \lambda^5}{g} \right). \quad (6.4)$$

The eddy vertical heat flux has a very steep dependence (fifth order) on  $\xi$  (when  $\lambda$  is fixed). [This is, of course, the same dependence that Held and Larichev (1996) found for the strength of the energy cycle  $\epsilon = d(\text{EKE})/dt$  because both are related through (A.7).] Expressed in terms of similar quantities, the horizontal heat flux varies as

$$\overline{v'\theta'} \sim \xi^4 \left( \frac{\Theta_0 \lambda^5 f_0 \beta^2}{gH} \right). \quad (6.5)$$

This varies only as the criticality to the fourth power, as might also be apparent from (6.2).

Our prediction for the vertical heat flux is tested in Fig. 15. The agreement is again fairly good, although deviations from the theory are harder to detect than in other figures because of the wide vertical range. Certainly, discrepancies from the theory are obvious for moderate and high criticality in the five sets of runs varying  $\tau$  (marked with circles in the figure); this is not surprising because the adiabatic eddy assumption is clearly violated for small  $\tau$ . In that limit, the mixing slope is flatter than the isentropic slope because not all the EAPE generated by the stirring is converted to eddy kinetic energy (some is damped diabatically).

A closure of the vertical heat flux such as (6.4) has important implications for the maintenance of the stratification and the difficulty of changing the criticality, as noted by Held (2007). Suppose that in a statistically

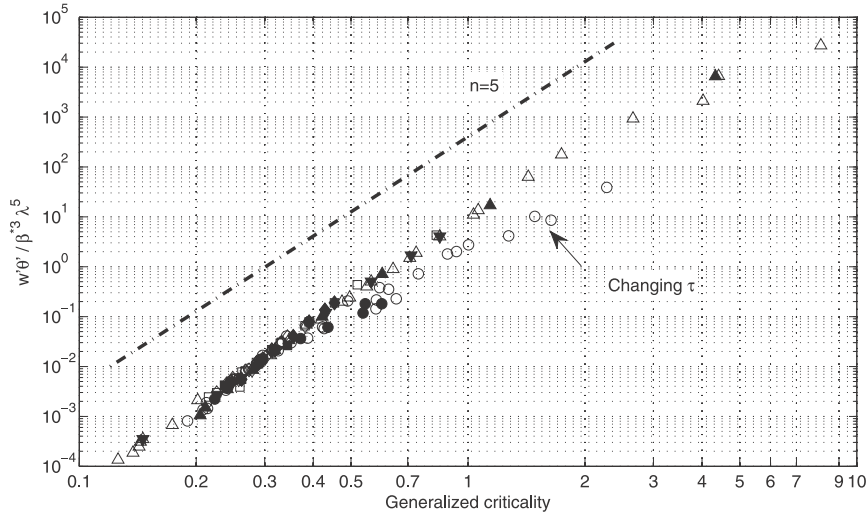


FIG. 15. As in Fig. 11, but for the normalized eddy vertical heat flux.

steady state there is a balance between the upward eddy fluxes of heat and some diabatic forcing  $\mathcal{H}_V$  that acts to destabilize the flow. That is, suppose that

$$\mathcal{H}_V \approx \frac{\partial w'\theta'}{\partial z}, \quad (6.6)$$

and so, using (6.4),

$$\mathcal{H}_V \sim \frac{\Theta_0}{gH} \beta^3 \lambda^5 \xi^5. \quad (6.7)$$

Suppose now that the forcing depends on the stratification only, in the form  $\mathcal{H}_V \sim \bar{\theta}_z^{-r}$ , where  $r$  is some unspecified exponent (a Newtonian relaxation to an unstratified state has  $r = 1$ ). Noting that  $\lambda \sim \bar{\theta}_z^{-1/2}$ , we obtain  $\bar{\theta}_y^5 \sim \bar{\theta}_z^{r+5/2}$ , omitting other scaling factors that are supposed constant in the present context. This expression may be alternatively expressed as

$$\xi \sim \bar{\theta}_y^{-(2r-5)/(2r+5)} \quad \text{or} \quad \xi \sim \bar{\theta}_z^{-r/5-1/2}. \quad (6.8)$$

These are expressions of the criticality in terms of the horizontal temperature gradient and the stability. They predict, other factors being constant, that as long as the dependence of the vertical heating on the stratification is not too strong ( $r < 5/2$ ), the isentropic slope should *decrease* both with increasing  $\bar{\theta}_z$  and with increasing  $\bar{\theta}_y$ ; that is, somewhat counterintuitively, the criticality should decrease with increasing lateral temperature gradient. Physically, the result stems from the fact that the vertical heat flux increases more rapidly than the horizontal heat flux with horizontal temperature gradient. If  $r = 1$  criticality varies very weakly with lateral temperature gradient, as the  $-3/7$  power, and if  $r = 2$  the

variation is even weaker,  $-1/9$ . These precise results are unlikely to hold in a primitive equation model or in the real world, but they do suggest a reason for the observed near-constancy of the isentropic slope with varying seasons, in which the external factor that varies most is the forcing of the horizontal temperature gradient. Our numerical experiments (with  $r \approx 1$ ) are qualitatively consistent with this prediction in that criticality does decrease with increasing  $\delta_y$  in our simulations. We will investigate in a subsequent study the sensitivity of the criticality on the forcing for other forms of forcing.

## 7. Conclusions

In this study we have explored the equilibration properties of a primitive equation model and a quasigeostrophic model, forced in similar ways and using the same stratification. This stratification is internally determined in the primitive equation model but prescribed in the quasigeostrophic model based on the primitive equation integrations.

Our first general result is that the mean flow and eddy statistics of the primitive equation two-level model were reasonably well predicted by the quasigeostrophic theory, over a range of parameters including those most representative of the earth's atmosphere, when the stratification is diagnosed from the PE model. We mean by "reasonably well" that the two models are nearly always qualitatively similar, and in many cases quantitatively similar—see Fig. 3 for example. To have a fully closed framework, a theory for the eddy vertical heat fluxes is also needed, and such a theory is available when the eddies are adiabatic, in which case the mixing slope is the isentropic slope. This leads to an estimate of

the eddy vertical heat flux in good agreement with the numerical results.

It may seem that the agreement between the primitive equation and quasigeostrophic models should not be surprising, but there is no guarantee that the dynamics of a PE model will remain in a quasigeostrophic regime. A consequence of the similarity is that various aspects of quasigeostrophic theory, in particular pertaining to the equilibration properties of baroclinic eddies, can be applied to the PE model. Thus, notions of the passage of energy to a barotropic state and an inverse cascade of barotropic energy halted by the beta effect are found to have relevance in a primitive equation model and so, potentially, in planetary atmospheres. The dependence of the eddy scales on criticality in our model was also in reasonable agreement with the QG scaling proposed by Held and Larichev (1996), except in one major regard: the eddy scales in our model are sensitive to friction, as is also found in some quasigeostrophic simulations (e.g., Thompson and Young 2007). We also find that the barotropization in our model tends to be weaker than predicted by the theory, and this may be related to the effects of friction.

The earth's atmosphere is not observed to have an inverse cascade of barotropic energy: certainly, no classical  $-5/3$  spectrum is observed, and the scales of the baroclinic eddies are little bigger than the deformation scale or the scale of the instability. This is also true in our control simulation, and the proximate reason for this is that the deformation radius that results from the model integration in that particular parameter regime is so large that no inverse cascade is possible. However, we do not find this to be a general result. We find the criticality to be a more or less continuous function of the parameters of the model rather than being tied to a marginally critical or subcritical value.

In particular, the criticality is a strong function of  $f_0/\beta$ , the diabatic time scale, and, notably, the factors affecting the stratification. Typically, we can obtain higher supercriticality by increasing  $f_0/\beta$  and the diabatic forcing rate and by reducing the stratification. In fact, we were able to change the model's criticality over one order of magnitude, and in some simulations in a large domain we were able to produce a robust inverse cascade with a  $-5/3$  spectrum.

The criticality, however, is not found to vary significantly with horizontal temperature gradient. This may be interpreted with the help of quasigeostrophic theory, which suggests that the vertical heat flux may increase even more rapidly than the horizontal heat flux does with horizontal temperature gradient, thus constraining the isentropic slope quite severely. The criticality will then vary quite weakly, although continuously, with

variations in meridional temperature gradient. This mechanism may provide a partial explanation as to the apparent near-constancy of the observed isentropic slope with season, where the main change in the forcing is the meridional gradient of incoming solar radiation.

All of our results should be tempered by the simplicity and in some aspects unrealism of our model, at least vis à vis the earth's atmosphere—notably the thermodynamic forcing scheme, the geometry, the limited vertical resolution, and, relatedly, the inexact conservation of potential vorticity and the effectively fixed tropopause height. The fixed tropopause potentially limits the ability of the model to fully respond to changes in forcing and is a topic we are currently investigating. Nevertheless, our results do suggest that the mechanisms determining the stratification may be important for allowing a rotating, stratified fluid to become supercritical. In our model the stratification is only weakly constrained by the slow Newtonian cooling, and there is no convective parameterization (although grid-scale convection does occur in places). A convective scheme that acts when the stratification is weaker than some threshold could, when used in conjunction with a relatively slow Newtonian cooling, convectively stabilize rapidly as needed, thus increasing the average deformation radius and producing a less baroclinically supercritical state, possibly imposing a maximum criticality that depends on the lapse rate of choice. The real atmosphere is more strongly destabilized in the vertical but has strong intermittent convection and a tropopause that can potentially vary in height, so the constraints of the stratification and the criticality are likely to be different again. Understanding how baroclinic eddies equilibrate in more realistic models that try to mimic these effects is both an intellectual and computational challenge.

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## APPENDIX A

### Description of the Primitive Equation Model

We have configured the MIT GCM to model a two-level Boussinesq fluid on the beta plane. The MIT GCM

is a gridpoint model with an Arakawa C grid and a Lorenz discretization in the vertical direction. Figure A1 illustrates the levels at which variables are calculated in our two-level version. Based on this discretization, the horizontal momentum, continuity, hydrostatic, and thermodynamic equations adopt the following forms:

$$\frac{\partial \mathbf{v}_k}{\partial t} + \mathbf{v}_k \cdot \nabla \mathbf{v}_k + \frac{1}{H} w(\mathbf{v}_2 - \mathbf{v}_1) + f \hat{\mathbf{k}} \times \mathbf{v}_k + \frac{1}{\rho_0} \nabla p_k = -\nu \nabla^4 \mathbf{v}_k - \delta k_1 \frac{\mathbf{v}_k}{\tau_F}, \quad (\text{A.1})$$

$$2w = -H \nabla \cdot \mathbf{v}_1 = H \nabla \cdot \mathbf{v}_2, \quad (\text{A.2})$$

$$4(p_2 - p_1) = (b_1 + b_2) \rho_0 H, \quad (\text{A.3})$$

$$\frac{\partial \theta_k}{\partial t} + \mathbf{v}_k \cdot \nabla \theta_k + \frac{1}{H} w(\theta_2 - \theta_1) = -\frac{1}{\tau} (\theta_k - \theta_{Rk}), \quad (\text{A.4})$$

where  $k = 1$  ( $k = 2$ ) for the lower (upper) level;  $\mathbf{v}_k$  is the horizontal velocity vector, and the gradient operator is likewise two-dimensional;  $\theta_k$ ,  $b_k$ , and  $p_k$  are the potential temperature, buoyancy, and pressure at each level, respectively; and  $\theta_{Rk}$  is the radiative equilibrium potential temperature. A nonzero vertical velocity  $w$  is only defined at midlevel;  $H$  is the total fluid depth and the Coriolis parameter varies meridionally according to the expression  $f = f_0 + \beta_y$ .

As discussed in the introduction, our aim is to produce a minimal primitive equation model that is as similar as possible to the standard two-layer quasigeostrophic model. With this in mind, we ignore compressibility effects and use the same reference density  $\rho_0 = 1 \text{ kg m}^{-3}$  in both levels. Buoyancy is calculated using the following linear equation of state:

$$b = -g \frac{\rho - \rho_0}{\rho_0} = g \alpha_\theta (\theta - \Theta_0) = g \frac{\theta - \Theta_0}{\Theta_0}. \quad (\text{A.5})$$

This equation was obtained by linearizing the ideal gas law  $\rho = p/R\Pi\theta$  around  $\theta = \Theta_0$ , which gives a thermal expansion coefficient  $\alpha_\theta = 1/\Theta_0 = 3.33 \times 10^{-3} \text{ K}^{-1}$  for  $\Theta_0 = 293 \text{ K}$ . Note that temperature and potential temperature are equivalent for our fluid because density only depends on temperature.

Combining the hydrostatic and geostrophic balances in the usual way leads to the following thermal wind equation in our model:

$$\nabla(\theta_1 + \theta_2) = -\frac{4f\Theta_0}{gH} \hat{\mathbf{k}} \times (\mathbf{v}_2 - \mathbf{v}_1). \quad (\text{A.6})$$

Hence, only the vertically averaged temperature is constrained by balance, whereas the discrete static stability  $\Delta\theta = \theta_2 - \theta_1$  and the potential temperature at each level are not.

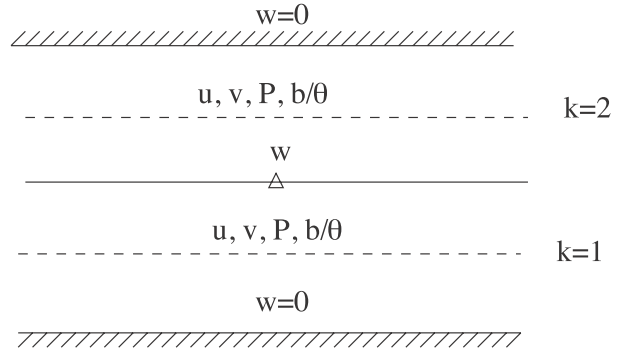


FIG. A1. Sketch of the vertical discretization, indicating the locations where variables are calculated.

It is also easy to show that this model conserves energy in the unforced, inviscid limit:

$$\frac{d}{dt} \text{KE} = \frac{1}{2} \frac{g}{\Theta_0} \langle w(\theta_1 + \theta_2) \rangle = -\frac{d}{dt} \text{APE}, \quad (\text{A.7})$$

where kinetic and available potential energies are defined as follows:

$$\text{KE} = \sum_{k=1,2} \frac{1}{2} \langle u_k^2 + v_k^2 \rangle, \quad (\text{A.8})$$

$$\text{APE} = \frac{1}{4} \frac{gH}{\Theta_0} \frac{1}{\langle \theta_2 - \theta_1 \rangle} \sum_{k=1,2} \langle (\theta_k - \langle \theta_k \rangle)^2 \rangle, \quad (\text{A.9})$$

and the angle operator  $\langle \cdot \rangle = \iint dx \, dy$  denotes a horizontal average.

Using this expression, we can write the eddy available potential energy as

$$\begin{aligned} \text{EAPE} &= \frac{1}{4} \frac{gH}{\Theta_0} \frac{1}{\langle \theta_2 - \theta_1 \rangle} \langle \theta_1'^2 + \theta_2'^2 \rangle \\ &= \frac{1}{8} \frac{gH}{\Theta_0} \frac{1}{\langle \theta_2 - \theta_1 \rangle} \langle (\theta_1' + \theta_2')^2 + (\theta_1' - \theta_2')^2 \rangle. \end{aligned} \quad (\text{A.10})$$

We can see that EAPE is made up of two components, associated with perturbations in either the vertically averaged temperature or the discrete static stability. As noted in (A.6), only the first of these two terms is in balance with the baroclinic wind. Hence, we use only this component to estimate the extension of the inverse cascade (see appendix B).

On the other hand, this model does not have a conserved potential vorticity. Quasigeostrophic potential vorticity is not conserved because this is not a quasigeostrophic model, and Ertel's potential vorticity is not conserved because of finite differencing, especially in the vertical. (It is rare that Ertel potential vorticity is

conserved in a PE model.) Because potential vorticity is not conserved, there is no exact Charney–Stern criterion. The criticality defined by (2.6) for this model is based on a generalization of its continuous counterpart and does not necessarily imply an exact connection to instability.

Finally, we point out that although the formulation presented above assumes a rigid lid boundary condition at the model's top (as in the standard quasigeostrophic two-layer model), in practice most of our simulations were calculated using a linearized free surface boundary condition, better posed numerically. This modification has little impact on the results because the external deformation radius is large, and the typical variations in surface height are much smaller than the layer depth. This was also explicitly checked by comparing simulations with both boundary conditions for some cases.

## APPENDIX B

### Inverse Cascade Estimates

In this appendix we discuss the relation between two different estimates of the extension of the inverse cascade: the ratio between the eddy scale  $L$  and the Rossby radius  $\lambda$ , and the ratio between EAPE and BCEKE in the general (continuous) PE problem. Using the Bousinesq variants of Lorenz's approximate expressions (e.g., Vallis 2006), we express EAPE as the global integral of the density:

$$\epsilon_\theta = \frac{b'^2}{\bar{b}_z} = \frac{g^2 \theta'^2}{\Theta_0^2 N^2}, \quad (\text{B.1})$$

where  $b$  is buoyancy and  $b' = -g\rho'/\rho_0 = +g\theta'/\Theta_0$ .

Expressing the temperature anomalies as  $\theta' \sim L\partial_y\theta'$ , where  $L$  is a characteristic eddy scale, and using the thermal wind constraint  $\partial_y\theta' = -f_0\Theta_0\Delta u'/(gH)$ , where  $\Delta u'$  is a characteristic eddy baroclinic velocity, this may be rewritten as

$$\epsilon_\theta = \left(\frac{L}{\lambda}\right)^2 \Delta u'^2 = \left(\frac{L}{\lambda}\right)^2 \epsilon_{\Delta u}, \quad (\text{B.2})$$

where  $\epsilon_{\Delta u}$  is the density for the baroclinic eddy kinetic energy. We assumed that the baroclinic eddy kinetic energy is dominated by modes of tropospheric depth (which would be related by balance to deep potential temperature anomalies or anomalies in the vertically integrated temperature). The ratio between EAPE and BCEKE scales as  $(L/\lambda)^2$  because thermal anomalies are created by stirring of the temperature gradients over length scales  $L$  that may be larger than the Rossby radius  $\lambda$ .

A key assumption in the above derivation is that the spatial scale of the thermal anomalies is the eddy scale  $L$  (i.e.,  $\theta' \sim L\partial_y\theta'$ ). This is true for a tracer that is quasi-conserved by the Lagrangian motion but is not expected to hold for  $\theta$  in the limit of strong diabatic forcing. In that limit, the spatial scale of the thermal anomalies should be smaller than the Lagrangian eddy scale (e.g., Swanson and Pierrehumbert 1997). As a result, the EAPE to BCEKE ratio is not an appropriate estimate of the inverse cascade for strong forcing. This may explain the erratic strong-forcing behavior in the simulations of Zurita-Gotor (2008), who found discrepancies in the dependencies of both ratios. The problem is further aggravated in that study by the use of a different time scale for eddies and mean, which effectively uncouples the heating from the Lagrangian motion.

Likewise, balance is another important consideration. In our two-level model, there is only one baroclinic mode, whose kinetic energy is related via thermal wind balance (A.6) to anomalies in the vertically averaged potential temperature. However, there is an additional contribution to EAPE [second term in (A.10)] that is not constrained by balance. Hence, only the first term in (A.10) and not the full EAPE should be included when using the EAPE/BCEKE ratio to estimate the extension of the inverse cascade in our two-level model.

## REFERENCES

- Balasubramanian, G., and S. T. Garner, 1997: The role of momentum fluxes in shaping the life cycle of a baroclinic wave. *J. Atmos. Sci.*, **54**, 510–533.
- Chang, E. K. M., 1993: Downstream development of baroclinic waves as inferred from regression analysis. *J. Atmos. Sci.*, **50**, 2038–2053.
- Charney, J. G., 1971: Geostrophic turbulence. *J. Atmos. Sci.*, **28**, 1087–1095.
- Chen, G., I. M. Held, and W. Robinson, 2007: Sensitivity of the jet latitude to friction. *J. Atmos. Sci.*, **64**, 2899–2915.
- Farrell, B. F., 1989: Optimal excitation of baroclinic waves. *J. Atmos. Sci.*, **46**, 1193–1206.
- Gage, K. S., and G. D. Nastrom, 1986: Theoretical interpretation of atmospheric wavenumber spectra of wind and temperature observed by commercial aircraft during GASP. *J. Atmos. Sci.*, **43**, 729–740.
- Green, J. S. A., 1970: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **96**, 157–185.
- Gutowski, W. J., Jr., 1985: Baroclinic adjustment and midlatitude temperature profiles. *J. Atmos. Sci.*, **42**, 1733–1745.
- , L. E. Branscome, and D. A. Stewart, 1989: Mean flow adjustment during lifecycles of baroclinic waves. *J. Atmos. Sci.*, **46**, 1724–1737.
- Held, I. M., 1978: The vertical scale of an unstable baroclinic wave and its importance for eddy heat flux parameterizations. *J. Atmos. Sci.*, **35**, 572–576.
- , 2007: Progress and problems in large-scale atmospheric dynamics. *The Global Circulation of the Atmosphere*,

- T. Schneider and A. H. Sobel, Eds., Princeton University Press, 1–21.
- , and M. J. Suarez, 1994: A proposal for the intercomparison of the dynamical cores of atmospheric general circulation models. *Bull. Amer. Meteor. Soc.*, **75**, 1825–1830.
- , and V. D. Larichev, 1996: A scaling theory for horizontally homogeneous, baroclinically unstable flow on a beta plane. *J. Atmos. Sci.*, **53**, 946–952.
- James, I. N., 1987: Suppression of baroclinic instability in horizontally sheared flows. *J. Atmos. Sci.*, **44**, 3710–3720.
- Juckes, M. N., 2000: The static stability of the midlatitude troposphere: The relevance of moisture. *J. Atmos. Sci.*, **57**, 3050–3057.
- Lapeyre, G., and I. M. Held, 2003: Diffusivity, kinetic energy dissipation, and closure theories for the poleward eddy heat flux. *J. Atmos. Sci.*, **60**, 2907–2916.
- Larichev, V. D., and I. M. Held, 1995: Eddy amplitudes and fluxes in a homogeneous model of fully developed baroclinic instability. *J. Phys. Oceanogr.*, **25**, 2285–2297.
- Maltrud, M. E., and G. K. Vallis, 1991: Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence. *J. Fluid Mech.*, **228**, 321–342.
- Marshall, J., A. Adcroft, C. Hill, L. Perelman, and C. Heisey, 1997: A finite-volume, incompressible Navier–Stokes model for studies of the ocean on parallel computers. *J. Geophys. Res.*, **102**, 5753–5766.
- O’Gorman, P., and T. Schneider, 2008: Weather-layer dynamics of baroclinic eddies and multiple jets in an idealized general circulation model. *J. Atmos. Sci.*, **65**, 524–535.
- Ollitrault, M., C. Gabillet, and A. Colin de Verdière, 2005: Open ocean regimes of relative dispersion. *J. Fluid Mech.*, **533**, 381–407.
- Pavan, V., and I. M. Held, 1996: The diffusive approximation for eddy fluxes in baroclinically unstable jets. *J. Atmos. Sci.*, **52**, 1262–1272.
- Rhines, P. B., 1975: Waves and turbulence on a  $\beta$ -plane. *J. Fluid Mech.*, **69**, 417–443.
- , 1977: The dynamics of unsteady currents. *The Sea*, E. D. Goldberg et al., Eds., Marine Modeling, Vol. 6, J. Wiley and Sons, 189–318.
- Salmon, R., 1978: Two-layer quasi-geostrophic turbulence in a simple special case. *Geophys. Astrophys. Fluid Dyn.*, **10**, 25–52.
- , 1980: Baroclinic instability and geostrophic turbulence. *Geophys. Astrophys. Fluid Dyn.*, **10**, 25–52.
- Schneider, T., 2004: The tropopause and thermal stratification in the extratropics of a dry atmosphere. *J. Atmos. Sci.*, **61**, 1317–1340.
- , and C. C. Walker, 2006: Self-organization of atmospheric macroturbulence into critical states of weak nonlinear eddy–eddy interactions. *J. Atmos. Sci.*, **63**, 1569–1586.
- Scott, R. B., and F. Wang, 2005: Direct evidence of an oceanic inverse kinetic energy cascade from satellite altimetry. *J. Phys. Oceanogr.*, **35**, 1650–1666.
- Simmons, A., and B. Hoskins, 1978: The life cycles of some nonlinear baroclinic waves. *J. Atmos. Sci.*, **35**, 414–432.
- Stammer, D., 1997: Global characteristics of ocean variability estimated from regional TOPEX/Poseidon altimeter measurements. *J. Phys. Oceanogr.*, **27**, 1743–1769.
- Stone, P. H., 1972: A simplified radiative–dynamical model for the static stability of rotating atmospheres. *J. Atmos. Sci.*, **29**, 405–418.
- , 1978: Baroclinic adjustment. *J. Atmos. Sci.*, **35**, 561–571.
- , and L. Branscome, 1992: Diabatically forced, nearly inviscid eddy regimes. *J. Atmos. Sci.*, **49**, 355–367.
- Swanson, K., and R. T. Pierrehumbert, 1997: Lower-tropospheric heat transport in the Pacific storm track. *J. Atmos. Sci.*, **54**, 1533–1543.
- Thompson, A. F., and W. Young, 2007: Two-layer baroclinic eddy heat fluxes: Zonal flows and energy balance. *J. Atmos. Sci.*, **64**, 3214–3231.
- Vallis, G. K., 1988: Numerical studies of eddy transport properties in eddy-resolving and parameterized models. *Quart. J. Roy. Meteor. Soc.*, **114**, 183–204.
- , 2006: *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, 745 pp.
- , and M. E. Maltrud, 1993: Generation of mean flows and jets on a beta plane and over topography. *J. Phys. Oceanogr.*, **23**, 1346–1362.
- Zhou, S., and P. H. Stone, 1993a: The role of large-scale eddies in the climate equilibrium. Part I: Fixed static stability. *J. Climate*, **6**, 985–1001.
- , and —, 1993b: The role of large-scale eddies in the climate equilibrium. Part II: Variable static stability. *J. Climate*, **6**, 1871–1881.
- Zurita-Gotor, P., 2007: The relation between baroclinic adjustment and turbulent diffusion in the two-layer model. *J. Atmos. Sci.*, **64**, 1284–1300.
- , 2008: The sensitivity of the isentropic slope in a primitive equation dry model. *J. Atmos. Sci.*, **65**, 43–65.
- , and R. S. Lindzen, 2006: A generalized momentum framework for looking at baroclinic circulations. *J. Atmos. Sci.*, **63**, 2036–2055.