

Emergence of Fofonoff states in inviscid and viscous ocean circulation models

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ABSTRACT

Numerical experiments are performed to directly test the emergence of the Fofonoff solution in an inviscid closed barotropic domain, and to explore its significance to the weakly dissipative system. The Fofonoff solution, characterized by a linear relationship between absolute vorticity and streamfunction, is generally realized as the time mean state of inviscid simulations over a fairly broad parameter range of varying (β -plane) Rossby number and resolution, in different geometrical domains, and with and without topography. The relevance of the Fofonoff solution to the viscous, decaying system is examined by numerical experiments with two different forms of viscosity, namely, biharmonic and harmonic, as well as with various boundary conditions. It is found that the boundary condition is generally more important than the order of the viscosity in determining the time mean fields. All of the frictional forms and boundary conditions prevented the complete realization of the Fofonoff state to a greater or lesser extent. Of the various boundary conditions used, the super-slip condition is most conducive to realizing a Fofonoff state. In this case, at high enough resolution the timescale of energy variability is much longer than a dynamical timescale, and the Fofonoff flow may be considered a ‘minimum enstrophy’ state. At high Reynolds number and high Rossby number an almost linear $q - \psi$ relationship can be achieved. For lower Rossby numbers, absolute vorticity tends to become homogenized, preventing the Fofonoff solution from arising. In the case of a free slip condition, it is still harder to reach a quasi-equilibrium. The time mean fields, after spin-up, generally show a two-gyre structure with homogenization in the absolute vorticity fields. In the no slip case, neither a quasi-equilibrium nor any well formed time mean field can be reached. As a slight generalization of the flow on β -plane, the inviscid topographic experiments also ultimately yield a linear relationship between absolute vorticity and streamfunction.

1. Introduction

Quasi-geostrophic flow in a single-layer on the β -plane is a common starting point for many dynamical studies in meteorology and oceanography. With forcing F and dissipation D the governing equation is

$$\frac{\partial \zeta}{\partial t} + J(\psi, q) = F + D, \quad (1.1)$$

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where, ψ is streamfunction, ζ is the relative vorticity given by $\zeta = \nabla^2\psi$, q is the potential vorticity defined by $q = \zeta + \beta y$ with β being the planetary vorticity gradient, and $J(\psi, q)$ is the Jacobian defined by $J(\psi, q) \equiv \psi_x q_y - q_x \psi_y$, where (x, y) are Cartesian coordinates in the (east, north) directions.

Two traditional and well tried methods of gleaning understanding from this important equation are an examination of steady (possibly forced dissipative) flows, and linearization about such steady solutions. At the other extreme, the inviscid dynamics of the full system can also reveal much about the system, especially its important 'free behavior.' It is with such behavior we shall be concerned in this paper. In addition, we shall be concerned with how the addition of a small viscous term, with the attendant necessity of additional boundary conditions, modifies the free behavior in a closed domain.

The flow governed by the unforced, inviscid version of (1.1), with the boundary condition of no normal flow on the boundary Γ , conserves total kinetic energy $E = \int_S \frac{1}{2}(\nabla\psi)^2 dx dy$, circulation $Q_1 = \int_S q dx dy$ and potential enstrophy $Q_2 = \int_S \frac{1}{2} q^2 dx dy$, where the integrals are taken over the whole domain S . Furthermore, the system is Liouvillean, and hence an ensemble of truncated or gridded versions of Eq. (1.1) may be expected to approach a thermal equilibrium state of maximum entropy (Kraichnan, 1975; Salmon *et al.*, 1976; Bennett and Middleton, 1983; Holloway, 1986a, b). The maximum entropy state is the most probable state in the phase space defined by the given values of the known invariants. If a single system explores all the allowable phase space and its time average is the maximum entropy state, the system is said to be ergodic. For most systems, it is very difficult to prove *a priori* whether the ergodic property will hold.

In a homogeneous situation (e.g. infinite or doubly-periodic domain with no topography) there can be no mean flow other than a purely zonal one. The spectrum of the eddy kinetic energy is then predicted to be

$$E(k) = \frac{\pi k}{\alpha(\mu + k^2)}, \quad (1.2)$$

where μ and α are constants determined from the values of the energy E , potential enstrophy Q_2 and $k_{\max}$ the maximum wavenumber, and k is the horizontal wavenumber. In an inhomogeneous environment there will be a steady component to the flow, and the resulting maximum entropy prediction for this is given by

$$\langle q \rangle = \mu \langle \psi \rangle + \lambda \quad (1.3a)$$

or

$$\nabla^2 \langle \psi \rangle + \beta y = \mu \langle \psi \rangle + \lambda, \quad (1.3b)$$

where λ is another constant. Prior to the realization of its possible statistical

mechanical significance, Fofonoff (1954, 1962) studied the analytical solution of Eq. (1.3) in a square basin with no-normal flow boundary condition; the solution to (1.3) is often referred to as the Fofonoff solution. Its putative importance stems from the fact that (1.3) exhibits a functional relationship between streamfunction and potential vorticity, and hence is an exact solution to the unforced, inviscid, equation of motion. [In a general sense, inertial, steady (at least in a statistical sense) mean flows are often referred to as *Fofonoff flows*.] A functional relationship of *any* form is a solution; in this paper, we shall use the term ‘Fofonoff solution’ in the restricted sense of a linear relationship between q and ψ . In statistical mechanical theory, a linear relationship arises when, out of the infinity of potential vorticity invariants, the statistical mechanics is constructed only from the quadratic invariants, potential enstrophy and energy, plus circulation. That is, the system is free to move anywhere on the phase surface of given energy E , circulation Q_1 and potential enstrophy Q_2 and the higher order constraints are neglected. This is in some ways an arbitrary choice (see e.g. Carnevale and Frederikson, 1987). If the projection of the higher order invariants on the energy-enstrophy surface is fairly uniform, this nevertheless may not be a poor way to construct the ensemble. Furthermore, any numerical model of any finite resolution, spectral or gridded, will normally conserve exactly only the quadratic invariants and therefore can only approach the equilibrium given by (1.2) and (1.3). However, it should be understood that the statistical mechanical theory constructed this way does build in a linear relationship between ψ and q . In the absence of a rigorous reason to build the ensemble this way, this remains somewhat ad hoc. A statistical mechanics can be constructed formally incorporating all the constraints (Miller, 1990), but this construction has not proven to be as practically useful, in a predictive sense, as the simpler theory (although see Robert and Sommeria, 1991).

With positive μ , which corresponds to a stable solution, the contour plots of a typical Fofonoff solution are shown in Figure 1. The absolute vorticity field is parallel to the streamfunction field, as required by the linear relationship. The relative vorticity is confined to the boundary layer, the thickness of the boundary layer l is given by

$$l \sim \frac{1}{\sqrt{\mu}} \sim R_o^{2/3} L, \quad (1.4)$$

where R_o is the β -plane Rossby number, defined as $R_o \equiv U_{\text{rms}}/\beta L^2$, where U_{rms} is the root mean square velocity, and L is the basin size. The boundary layer gets thinner as β increases, or total energy decreases. The absolute vorticity field is dominated by the planetary vorticity βy inside the basin, where the flow is westward. The flow returns to the eastern boundary through northern and southern boundary layers, forming two gyres, anticyclonic in the northern basin, cyclonic in southern basin. The parameter λ affects the symmetry of the fields: with zero λ , the fields are symmetrical

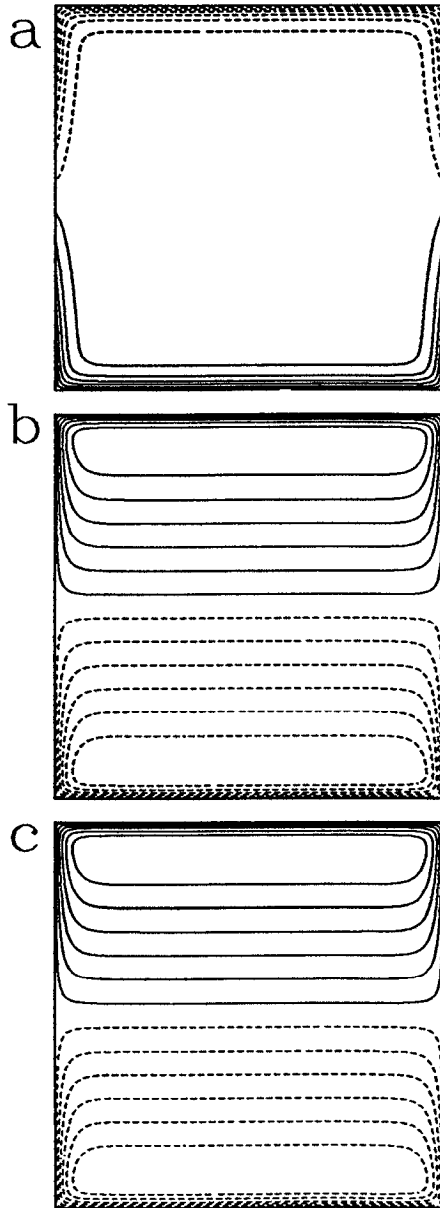


Figure 1. Exact Fofonoff solution in a $[-\pi, \pi] \times [-\pi, \pi]$ domain. The parameters μ , β and λ , defined in (1.3b), are 40, 10 and 0 respectively. Shown here are (a) relative vorticity ζ , (b) absolute vorticity q and (c) streamfunction ψ .

about $y = 0$; for general non-zero λ , one gyre will be enlarged, while the other will be squeezed, and in the extreme case one gyre will fill out the whole basin.

In the light of the above discussion, this paper seeks to answer the following questions: Is the Fofonoff solution in fact achieved as the time average state in a single realization of an integration of the equation of motion? As a corollary of this, how is this time mean Fofonoff solution actually achieved from random initial conditions? Second, how relevant is this time mean Fofonoff solution to the flow determined by (1.1) with the viscous terms restored. The first question is important simply as a fundamental question in its own right. The equilibrium theory per se says nothing about the approach to equilibrium, or whether it will be reached. In the absence of an H-theorem, one way to examine this problem is by direct numerical integration. (It should be noted that, according to information theoretic approaches to the subject, ergodicity properties are not required for the valid application of statistical mechanical methods to a system. See for example Buck and Macaulay, 1991.)

Even if (as will in fact be found) the inviscid system does approach a Fofonoff state, the *relevance* of such a solution to viscous circulation is of concern. Of course, the ocean does have an enormous Reynolds number ($\sim 10^{10}$ for the large scales) and a naïve scaling theory would discard viscous terms. However, vanishing viscosity is known to be a singular limit of the equations. Even in the absence of boundaries, viscosity is ultimately the only mechanism whereby energy and enstrophy can be dissipated; the turbulent cascade, and hence the spectrum of eddy energy, may therefore be expected to be profoundly altered by the omission of viscosity. Indeed, the smallest resolvable scale in a numerical simulation should be smaller than the viscous scale, and so a truncated model without somehow modelling dissipation is (as a model of a viscous fluid) inconsistent. On the other hand, in a closed domain the large-scale mean field may be less affected by viscosity and be close to a free solution. But now—since the order of the equation is increased by viscosity—the required additional boundary conditions may be important. However, this is a delicate point, since it may be possible to choose the boundary conditions in such a way as to minimize their effects on control of flow pattern. For example, in the classical steady Munk problem (see e.g. Hendershott, 1986) a fourth-order differential equation arises and boundary layers perforce arise on all walls. However, if a no stress (i.e. free-slip) condition, rather than a no-slip condition, is chosen at the zonal boundaries, boundary layers at these walls are not needed, and the interior flow is in any case little affected. Since the use of molecular friction is untenable, so-called ‘eddy viscosities’ are frequently used to parameterize the small scales. Because of their uncertain form, a prudent course may be to choose their form to have certain attractive properties, perhaps especially in minimizing the influence of artificial viscous boundary layers (Marshall, 1984). A simple system with which to explore

these question is the unforced version of (1.1), in which viscosity is a parameter to be varied. Thus, we compare the inviscid problem with a selection of viscous forms and boundary conditions to assess both the influence of boundary conditions and the like, as well as the relevance of Fofonoff solution to the viscous problem.

Another class of arguments of relevance to 'free' solutions to the equations of motion are so-called selective decay hypotheses (Bretherton and Haidvogel, 1976). The idea is that, in quasi-geostrophic systems, viscosity will preferentially dissipate enstrophy over energy. The end-state of a decaying, weakly forced system may therefore be a 'minimum enstrophy' state.² Consider arbitrary variations satisfying $\psi = 0$ on the boundary Γ , minimizing potential enstrophy Q_2 for given circulation Q_1 and energy E . We require

$$\delta \int_S \frac{1}{2} (\nabla^2 \psi + \beta y)^2 dx dy + \mu \delta \int_S \frac{1}{2} (\nabla \psi)^2 dx dy - \lambda \delta \int_S \nabla^2 \psi dx dy = 0. \quad (1.5)$$

After several integrations by parts this yields

$$\int_S \nabla^2 (\nabla^2 \psi + \beta y - \mu \psi) \delta \psi dx dy + \int_\Gamma (\nabla^2 \psi + \beta y - \lambda) \frac{\partial \delta \psi}{\partial n} ds = 0, \quad (1.6)$$

(where n is normal to the boundary) which gives, since both $\delta \psi$ and boundary value of $\partial \delta \psi / \partial n$ are arbitrary,

$$\nabla^2 (\nabla^2 \psi + \beta y - \mu \psi) = 0 \text{ within } S, \quad (1.7)$$

and

$$\nabla^2 \psi + \beta y - \lambda = 0 \text{ on } \Gamma. \quad (1.8)$$

Thus, using $\psi = 0$ on Γ , we obtain

$$\nabla^2 \psi + \beta y = \mu \psi + \lambda \text{ everywhere.} \quad (1.9)$$

Hence, minimization of potential enstrophy gives the same linear relationship between absolute vorticity and streamfunction as in (1.3). Indeed, Carnevale and Frederiksen (1987) demonstrated that in the limit of infinite resolution the maximum entropy state is identical to the minimum potential enstrophy state. Thus the Fofonoff solution may be realized in some weakly dissipative systems, for apparently quite different reasons than maximizing entropy in inviscid systems. Still, the evolution toward a minimum enstrophy state may of course be determined by nonlinear processes seeking to maximize entropy.

2. It should be pointed out that most viscosities preferentially dissipate relative enstrophy, not potential enstrophy, and the resulting state presumably should be a minimum relative enstrophy state, that can be described by (1.9) with $\beta = 0$, which is in general different from the minimum potential enstrophy state.

Numerical simulations of various nonlinear ocean circulation models have been noticed, under certain circumstances, to approach equilibrium states qualitatively resembling Fofonoff solutions (e.g. Veronis, 1966; see also Hendershott, 1986). Veronis solved Eq. (1.1) numerically with constant sinusoidal wind stress and bottom friction for a wide range of parameters which basically measure the relative importance of nonlinearity and viscosity. In his case (g), where nonlinearity is much more important than friction, the solution of ψ field looks rather similar to a Fofonoff solution. Griffa and Salmon (1989) and Cummins (1992) also saw the emergence of Fofonoff-like modes in weakly decaying simulations. To overcome a difficulty caused by piling up of energy at the smallest scales resolved by their model, Griffa and Salmon used the anticipated vorticity viscosity:

$$D_{AV} \equiv -\nu J(\psi, \nabla^2 J(\psi, \zeta)). \quad (1.10)$$

This scale-selective viscosity dissipates enstrophy $Z_2 \equiv \frac{1}{2} \int_S \zeta^2 \, dx dy$, but not energy or circulation. In implementing this eddy viscosity, they also used the boundary condition

$$\frac{\partial}{\partial n} J(\psi, \zeta) = 0 \quad (1.11)$$

in addition to the no normal flow condition $\psi = 0$. Cummins (1992) explored the effects of a more conventional viscosity.

By incorporating scale-selective viscosity in a numerical simulation one hopes that the effect of this is in some sense equivalent to an increase in resolution; by removing enstrophy and not energy, one seeks to mimic an infinite resolution model in which the eddy variability is pushed to smaller and smaller scales. However, a decaying simulation is not strictly a direct test of the theory, and using such a scale selective filter may evolve the system to the formally similar minimum enstrophy state. Since the theory of equilibrium statistical mechanics applies to a truncated model of any reasonable resolution that includes nonlinearity, one should expect that, given a long enough time average, a mean Fofonoff flow should obtain no matter what the resolution is. However if the resolution is too low, the nonlinear interactions may be insufficient to ensure ergodicity (e.g. Fox and Orszag, 1973). Further, at low resolution the large eddy variability may drown the signal for short averaging periods.

The plan of the rest of the paper is as follows: Numerical models are described in the next section. Section 3 presents simulations of the truly inviscid dynamics on the β -plane. Experimental results with different Rossby numbers, different resolution and stimulations in a triangular domain, are reported. We examine the roles of various forms of viscosity and different boundary conditions in Section 4. In Section 5 we present some results of flow over topography. Section 6 summarizes and concludes.

2. Numerical models

Eq. (1.1) with $F = 0$ is integrated using a gridded model. Lengths are non-dimensionalized such that, for the rectangular domain, the domain size is $2\pi \times 2\pi$, with resolution ranging from 32×32 to 256×256 grid points. The differential operations in Eq. (1.1) are approximated by centered, second-order-accurate, finite differences. A leapfrog time integral scheme is used with the occasional implementation of an Euler forward step to eliminate the consequences of time splitting. The Arakawa (1966) formulation is used for the Jacobian advection term. This exactly conserves energy and potential enstrophy, to the accuracy of the time-stepping scheme. Thus, although truncation errors are present, they do not destroy the quadratic invariants, and in this sense the numerical model is indeed inviscid, or at least ‘quadratically inviscid’. Higher order invariants are not exactly conserved, however.

In the inviscid simulations, the only boundary condition used is one of no normal flow. The boundary values of vorticity are updated, at each timestep, by direct integration using a one-sided form of the Arakawa Jacobian which maintains the integral invariants (Arakawa, 1966; Salmon and Talley, 1989). In the decaying experiments, the boundary values of relative vorticity are given by the imposed boundary conditions.

The random initial fields in the rectangular domain simulations are shown in Figure 2. The initial energy spectrum is concentrated between wavenumbers $k = 3$ and 4, and the total energy is chosen to be 15.55 (in non-dimensional units). The field is constrained to have $\int_S \zeta \, dxdy = \int_S q \, dxdy = 0$. In our simulations, β ranges from 1 to 10; the β -plane Rossby numbers are between 1.59×10^{-3} and 1.59×10^{-2} , the upper limit is the same as that of QG1 experiment in Griffa and Salmon (1989); that of their QG2 corresponds to the case with $\beta = 5$, and is perhaps closer to the realistic ocean values.

The initial condition for experiments in a triangular domain is similar, with total energy also 15.55. The resolution is 128×64 . We drop the requirement $\int_S q \, dxdy = 0$ since it depends on the value of β . The Fofonoff solution (if achieved) will be generally given by Eq. (1.3) with non-zero λ .

It is straightforward to generalize the numerical model on β plane to include topography; in fact, the β plane model can be regarded as a special case of the topographic model in that the general topographic term $h(x, y)$ in potential vorticity is replaced by βy . We describe the topographic experiments in more detail in Section 5.

Two types of scale selective viscosity are used in decaying experiments, one is harmonic (or Newtonian) viscosity:

$$D_h \equiv A_2 \nabla^2 \zeta; \quad (2.1)$$

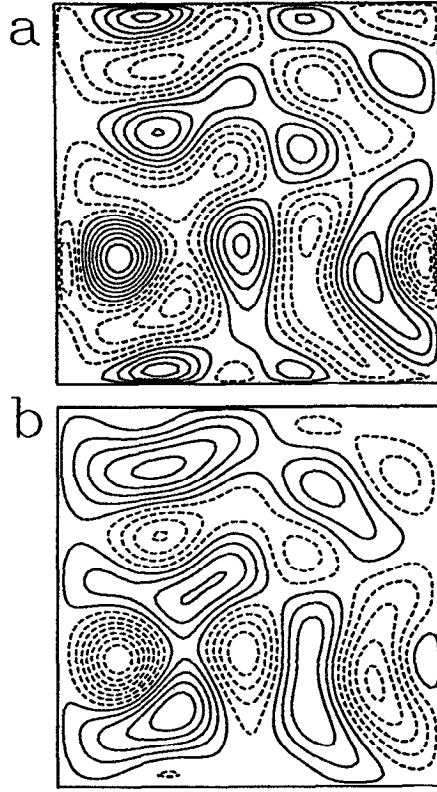


Figure 2. The initial random fields used in the experiments in rectangular basin. (a) is the relative vorticity ζ ; (b) streamfunction ψ .

the other is the biharmonic viscosity:

$$D_{bh} \equiv -A_4 \nabla^4 \zeta. \quad (2.2)$$

In addition to no-normal flow, one more boundary condition is required for harmonic viscosity, two more for biharmonic. Three different kinds of boundary condition are implemented in our decaying simulations. The first one is the so called superslip boundary condition given by:

$$\frac{\partial}{\partial n} \zeta = 0 \quad (2.3a)$$

and (for biharmonic viscosity)

$$\frac{\partial^3}{\partial^3 n} \zeta = 0. \quad (2.3b)$$

This boundary condition is not conventional; it is designed to minimize the effects of frictional boundary layers (Marshall, 1984). The secondary boundary condition implemented is the more usual free slip condition:

$$\zeta = 0, \quad (2.4a)$$

and (for biharmonic viscosity)

$$\frac{\partial^2}{\partial n^2} \zeta = 0. \quad (2.4b)$$

The biharmonic viscosity with boundary conditions (2.4) were adopted by Holland (1978), and have been widely used since. Circulation is not conserved. The third condition is the no-slip boundary condition:

$$\frac{\partial}{\partial n} \psi = 0, \quad (2.5)$$

it is implemented for harmonic viscosity only. Although it is the strictly correct condition for use with molecular viscosity, in ocean models it is often thought to be inappropriate since molecular boundary layers are not resolved.

In the inviscid simulations on β plane, one parameter which is found to be a useful indicator of approaching a Fofonoff solution is the anticorrelation function defined by

$$C \equiv - \int_S \zeta \beta y dx dy. \quad (2.6)$$

It measures the (anti)correlation between relative vorticity and latitude. For a random field, this correlation is zero. For the Fofonoff solution, the positive and negative relative vorticities are segregated to southern and northern part of the basin, and the function C is expected to be positive. Therefore if the Fofonoff solution emerges from random initial conditions, an increase in function C is anticipated.

Physically, since the potential enstrophy Q_2 is conserved, we have

$$\int_S \frac{1}{2} (\zeta + \beta y)^2 dx dy = \text{const}, \quad (2.7)$$

which gives

$$Z_2 - C = \text{const}. \quad (2.8)$$

In pure 2-D turbulence energy and enstrophy are both exactly conserved. Here with a non-zero β effect, $(Z_2 - C)$ rather than Z_2 is conserved, and Z_2 will tend to increase

if enstrophy cascades to small scales. This forces a negative correlation between vorticity and latitude. In the inviscid simulations, the anticorrelation function will experience a rapid increase in the early stage when the flow has a rapid adjustment to Fofonoff solution, and will then settle down to some constant, with some small oscillation, which corresponds to an average Fofonoff solution. In decaying simulation, although the potential enstrophy is not conserved, the anticorrelation function C is still a useful indicator of the achievement of a Fofonoff-like state.

3. Inviscid simulations

a. Experiments with different Rossby numbers. We first describe two experiments with different Rossby numbers in a rectangular domain. The initial energies are the same, the β -plane Rossby number in the first, ISRD1, is 1.59×10^{-2} and is 10 times larger than in ISRD2, 1.59×10^{-3} . The resolution is 64×64 for both experiments. Both experiments were run for extremely long times. Taking a turnover time T_e given by $T_e \equiv L/U_{\text{rms}}$ as a time unit, the experiments are run for about $90 T_e$, entailing an integration of more than several hundred thousand time steps. In this time period, the energy and potential enstrophy are conserved to within a factor of 10^{-6} or so. The anticorrelation functions are shown in Figure 3. They indicate that statistically-steady states are arrived after about $10 T_e$. After that, the anticorrelation functions simply oscillate around some constant, in a rather random way, and show no indication of ever dying. Also notice after approaching a constant, ISRD1 has larger relative oscillation amplitude.

Figure 4 shows a sequence of plots of streamfunction averaged in the first eddy turnover time, i.e. $0 - T_e$, and in 1 to $3 T_e$, 3 to $7 T_e$, and 7 to $11 T_e$ for ISRD2. In the early stage, the random eddies are pushed to the western boundary, then two gyres form right behind the northern and southern walls, which grow and fill out the northern and southern half of the basin. It is clear that a state close to Fofonoff solution is achieved after about $10 T_e$. For ISRD1, a similar trend is recognizable (if not quite as distinct.)

In ISRD2, the average fields of ψ , ζ and q clearly show that the fields have arrived at a Fofonoff solution. Small numerical noise, especially in the field of q , is eliminated (for presentation purposes only) by averaging the fields of the neighboring 9 grid points; the smoothed average fields are shown in Figure 5. It is obvious, from the plots, that the q field is parallel to ψ field, as required by $J(\psi, q) = 0$. In ISRD1, the fields are averaged over every time step after the first $5 T_e$. More noise seems to be present in q field here. In Figure 6 the smoothed average fields are shown. Here too the ψ field is almost the exact Fofonoff solution, and q field is close to parallel to the ψ field, although not as smooth. Figure 7a and 7b show the scatter plots of $q - \psi$ for ISRD2 and ISRD1 respectively, both q and ψ are smoothed average

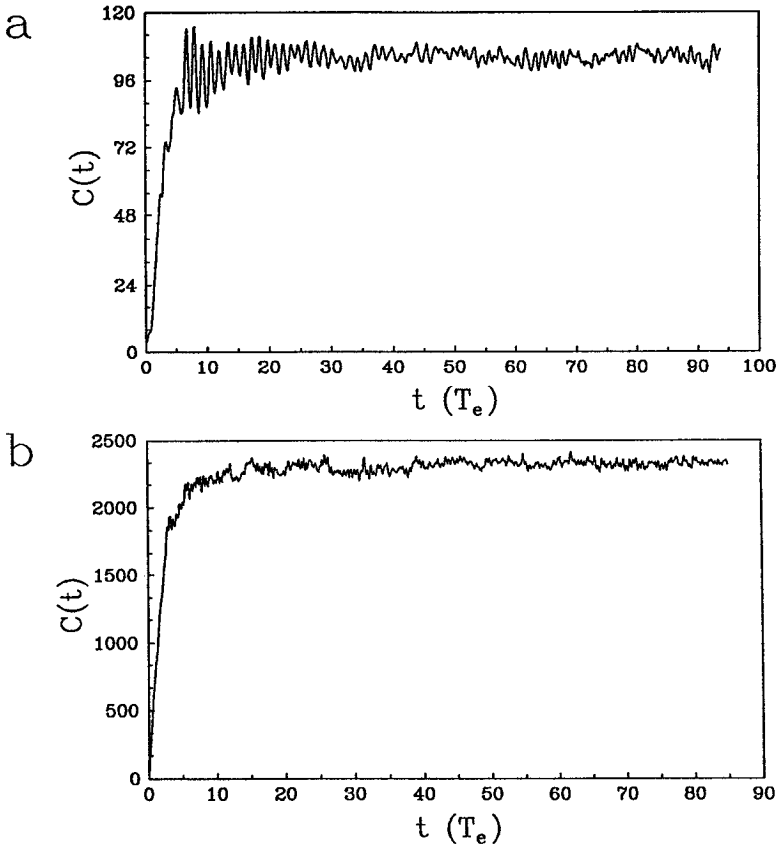


Figure 3. Time series of anticorrelation function in experiments (a) ISRD1 and (b) ISRD2, the unit of time is the eddy turnover time, $T_e = L/U_{rms}$. Experiments start from a random initial condition with no mean flow. The fields spin up after about $10 T_e$ in both cases.

values. Figure 7a shows a near-perfect linear relation, while Figure 7b shows a good functional relation, although it is not as linear as that in Figure 7a. Clearly, the time mean Fofonoff solution is realized in these two inviscid experiments.

The time-series of anticorrelation function and the scatter plot show that a linear $q - \psi$ relationship is easier to reach at smaller Rossby number. (On the other hand, we shall see that when viscosity is added simulations with a *large* Rossby number show a better linear relationship between q and ψ .) An explanation for this can be sought by assuming that Rossby waves are an effective mechanism of transmitting information, in analogy with the mid-latitude initial value problem of Anderson and Gill (1975). Starting from a zero field, then turning on the wind stress, they show that the field comprises steady and transient parts. With time, the transient moves westward at the speed of Rossby wave, ‘uncovering’ the steady Sverdrup flow. In our

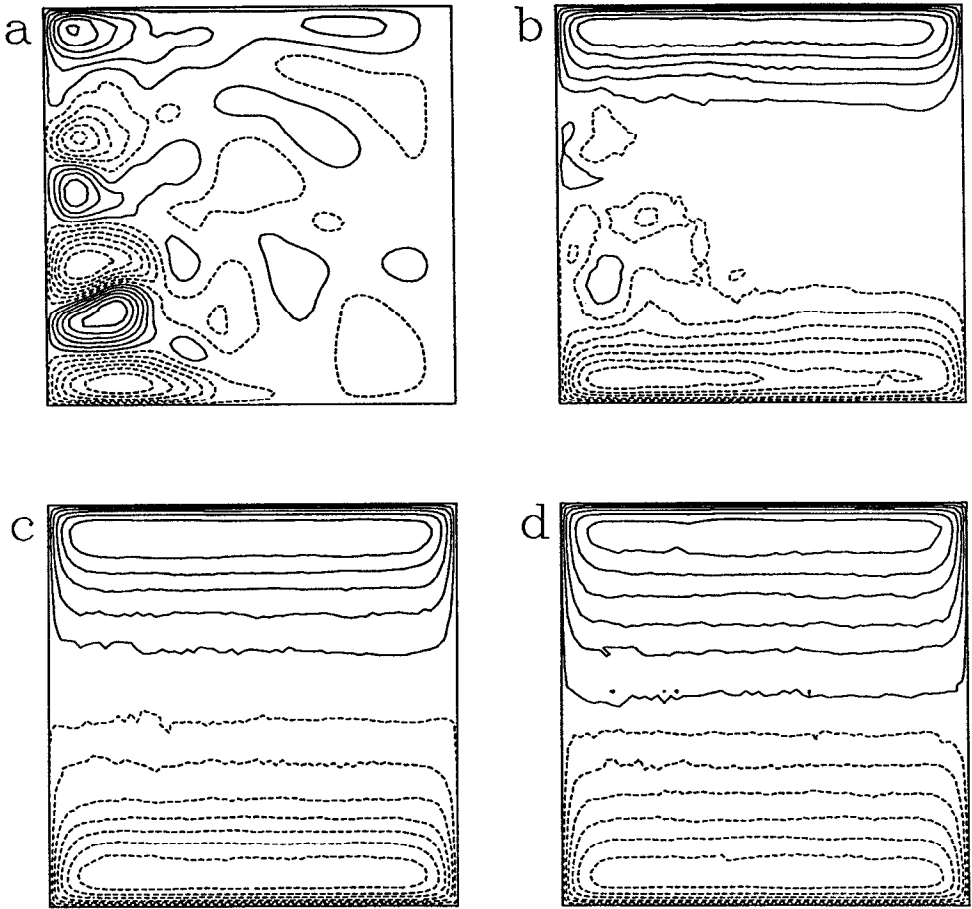


Figure 4. The early stage time evolution of the streamfunction ψ fields in experiment ISRD2. (a) is the ψ averaged every time step in the first T_e ; (b) averaged ψ field in the time interval 1–3 T_e ; (c) in 3–7 T_e ; (d) in 7–11 T_e .

case, the transient moves within the basin, uncovering the steady part, now the Fofonoff solution. For the Rossby wave to be effective, we must have

$$T_R \ll T_e, \quad (3.1)$$

where T_R is the time the Rossby wave takes to propagate from one side to another. Given the speed of the Rossby wave by $c_R = \beta/k^2$ (k is the wavenumber), we have, from (3.1)

$$T_R = \frac{L}{c_R} = \frac{Lk^2}{\beta} \ll \frac{L}{U_{\text{rms}}},$$

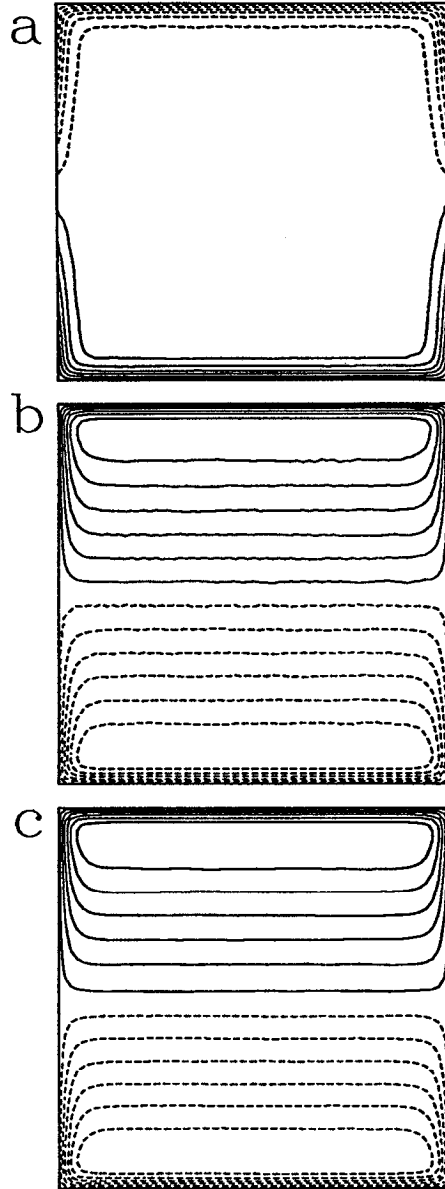


Figure 5. Time averages of (a) relative vorticity ζ ; (b) absolute vorticity q , and (c) streamfunction ψ for ISRD2. Averages are taken every time step for about $80 T_e$ after the first $7 T_e$ or so. Then the average fields shown are smoothed by averaging the neighboring 9-point fields to reduce the numerical noise.

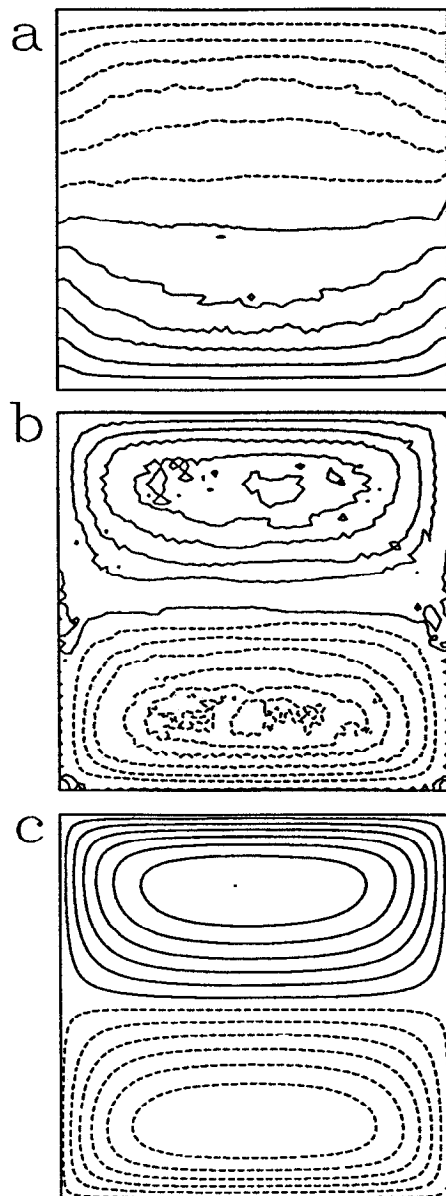


Figure 6. The smoothed time mean fields for ISRD1.

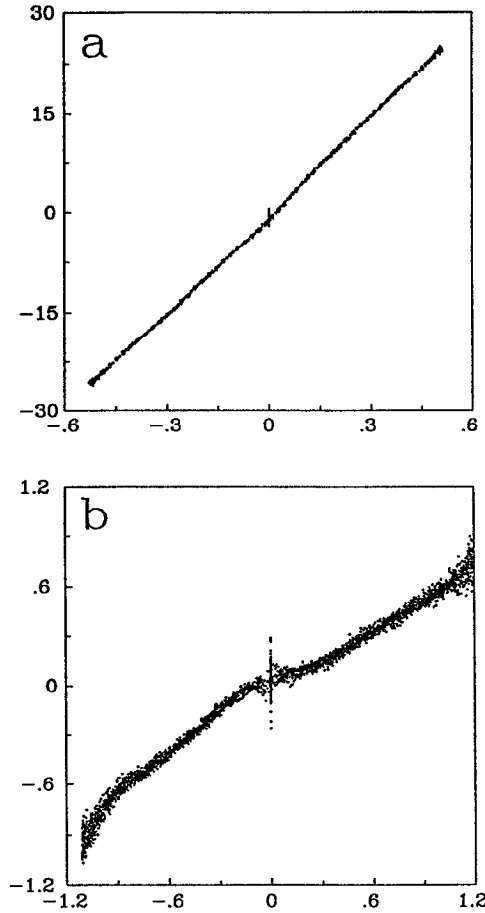


Figure 7. The scatter-plots of $q - \psi$ for experiments (a) ISRD2 and (b) ISRD1. Fields are smoothed time means, and q is plotted against ψ .

or

$$k \ll k_{\beta} = \left(\frac{\beta}{U_{\text{rms}}} \right)^{1/2}. \quad (3.2)$$

If the inequality $k \ll k_{\beta}$ can only be satisfied for scales larger than the domain, then Rossby waves will be a very inefficient mechanism for uncovering the time mean Fofonoff state. In ISRD1, k_{β} is around 1, i.e. the basin size scale; while in ISRD2, it is about 4. Thus, in the experiment with a smaller Rossby number (ISRD2) the Rossby waves are more efficient in producing a mean Fofonoff state.

Also, if initial Rossby number is high, the ratio of the potential enstrophy of the eddy motion to that of the mean Fofonoff state may be very large, leading to noticeable scatter in the $q - \psi$ plot for some finite time interval for averaging. Table 1

Table 1. Estimates of the ratios of energy and potential enstrophy of the fluctuating and mean (Fofonoff) components for ISRD1 and ISRD2. $\langle E \rangle$ and $\langle Q_2 \rangle$ are the energy and potential enstrophy of a Fofonoff solution with the same properties as the average state obtained in the simulations, and E' and Q'_2 are the residual energy and potential enstrophy.

Experiment	β	$E'/\langle E \rangle$	$Q'_2/\langle Q_2 \rangle$
ISRD1	1	~ 0.2	~ 48
ISRD2	10	~ 0.6	~ 0.4

shows the ratios of $E'/\langle E \rangle$ and $Q'_2/\langle Q_2 \rangle$, where $\langle E \rangle$ and $\langle Q_2 \rangle$ are the energy and potential enstrophy of an exact Fofonoff solution with the same properties as the average state of the simulation,³ and E' and Q'_2 are the residual energy and potential enstrophy (total minus that of Fofonoff solution). In the large Rossby number case, the initial potential enstrophy is much larger than the time mean Fofonoff solution, causing turbulent motion in the simulation. On the other hand, when R_o is small, the flow is dominated by the planetary vorticity which is more characteristic of Fofonoff solution. A straightforward test of this idea is to run the simulation with the smoothed time mean fields of ISRD1 as initial condition. Figure 8a–c shows the smoothed time mean fields from this simulation run for about an additional $10 T_e$. There is obvious improvement in the q field, evidently achieved by effectively filtering out most of turbulent eddy motion. Figure 8d, the corresponding scatter-plot, shows a much tighter relationship between q and ψ , although the deviation from linearity inherited from the earlier experiments remains.

b. Simulations at different resolutions. The parameters of the Fofonoff solution depend on, among other things, the resolution of the simulation. Carnevale and Frederiksen (1987) showed that the random eddy part will eventually go to zero as resolution goes to infinity, and the field at finite scales is merely time mean Fofonoff solution. Numerical simulations with higher resolution should therefore have a more energetic time-mean Fofonoff state than lower resolution simulation starting from the random initial condition with the same energy, reflected in a slight difference in the solution parameters.

Experiments with a resolutions 32×32 and 128×128 have been performed in a rectangular domain to study the variation of the time mean fields with the resolution. Both experiments have Rossby numbers 3.18×10^{-3} . The resulting fields are qualitatively the same. Careful analyses of the scatter-plots, however, show that the proportional constant μ is smaller in the higher resolution experiment. From (1.4), with β and L constants, the energy contained in the time mean Fofonoff flow is proportional to $1/\mu^{3/2}$, and hence increases as the resolution increases. This is

3. Three independent parameters are required to determine the Fofonoff solution. β is given, and the Fofonoff solution we are looking for is always symmetric, which determines λ . Hence only the linear constant μ in (1.3) is needed. This is obtained from the scatter-plot of $q - \psi$.

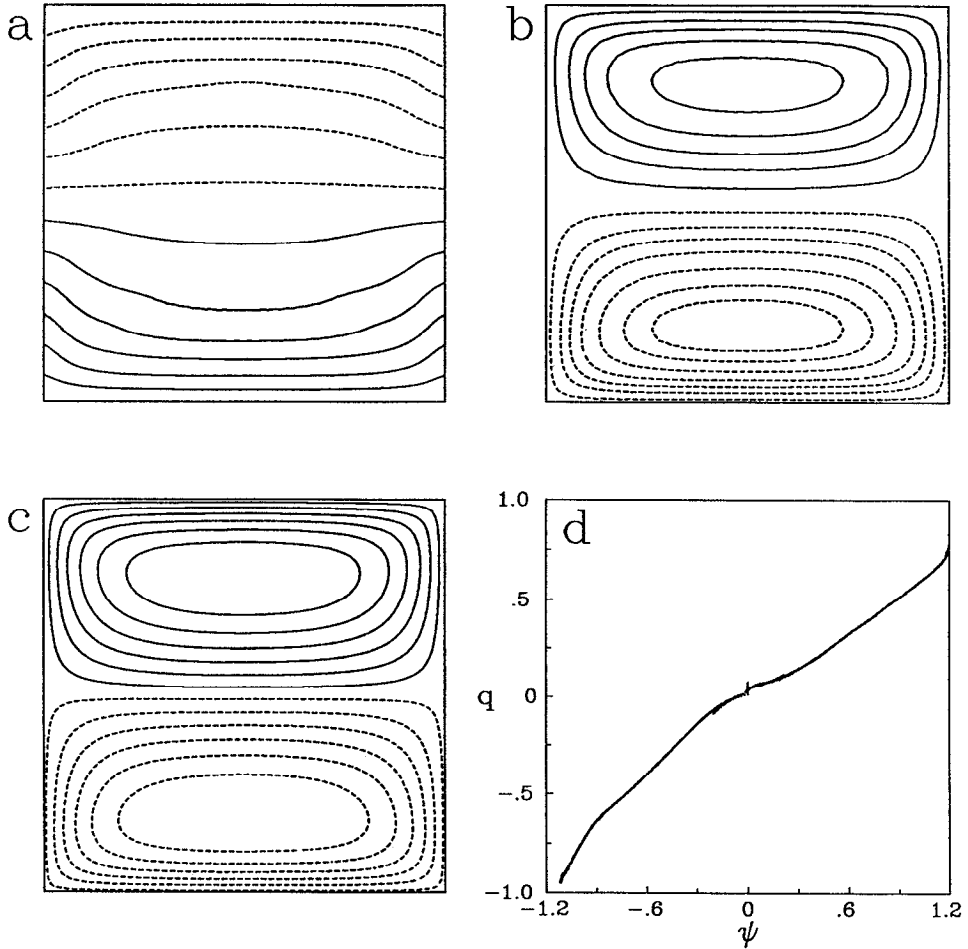


Figure 8. Experiment using the smoothed average ζ field in ISRD1 instead of random field as initial condition. Now the field has very little transient component. The fields are averaged from $10 T_e$, and then smoothed. Shown here are (a) relative vorticity ζ , (b) absolute vorticity q , (c) streamfunction ψ and (d) scatter-plot of $q - \psi$.

consistent with the theory. Roughly measuring from the scatter-plots, we obtain $\Delta\mu/\bar{\mu} \sim -0.37$ where $\bar{\mu}$ is the average of μ 's in the two experiments, $\Delta\mu$ is the increase in μ from 32×32 to 128×128 experiments. Therefore, we have:

$$\frac{\Delta E}{\bar{E}} = -\frac{3}{2} \frac{\Delta\mu}{\bar{\mu}} \sim 55.5\%.$$

Both energies are the components contained in the time mean fields. Hence, when $R_o = 3.18 \times 10^{-3}$ in the initial random flows, then as resolution is increased from

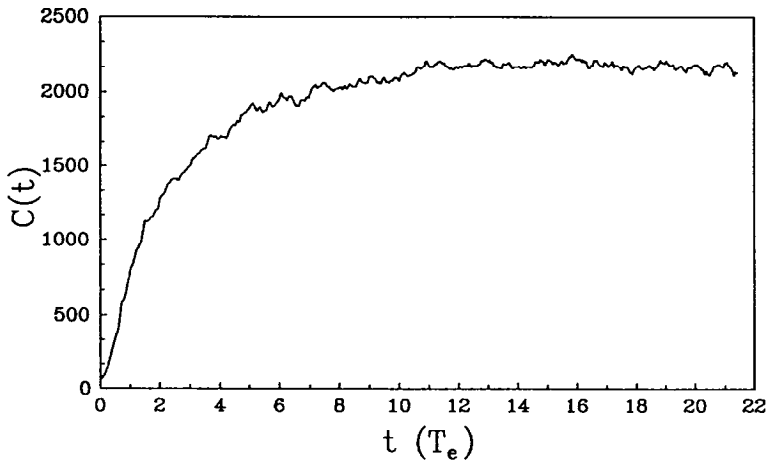


Figure 9. Time series of anticorrelation function in ISID2.

32×32 to 128×128 the relative increase of the steady component energy is more than half.

c. Time mean Fofonoff solution in irregular domains. As a straightforward generalization from the rectangular domain, the time mean Fofonoff solution in irregular domain is of some interest. The numerical code can be readily modified to any geometrical shape using the capacitance matrix algorithm (e.g. Pares-Sierra and Vallis, 1989), but we concentrate on a triangular domain. Clearly, real oceans are neither rectangular nor triangular, but doing the problem in a triangular domain will at least indicate if the rectangular domain is in any sense a special case. In fact, it is not.

The triangular domain is embedded in an rectangular domain of resolution 128×64 . We use a definition of Rossby number and turnover time based on the height of the domain. Two experiments ISID1 and ISID2 have been performed, with the only difference being the different β -plane Rossby numbers R_o used, 1.59×10^{-2} in ISID1 and 1.59×10^{-3} in ISID2.

In experiment ISID1, the fields spin up after about 6 turnover times T_e . However, the average fields for $20 T_e$ after the first $10 T_e$ do not indicate that a linear relation between ψ and q has been achieved, nor does the scatter plot. The ψ field has a nice two gyre structure, the one expected from the numerical solution of Eq. (1.3) in triangular domain (see Appendix). The ζ field is dominated in the boundary area, although the absolute vorticity field is too noisy to show any significant pattern.

Figure 9 shows the time series of anticorrelation function for $22 T_e$ in ISID2. It clearly indicates that an equilibrium state is arrived after the first $10 T_e$ or so. Figure 10 shows the smoothed time mean fields taken every time step after the first $10 T_e$ for $12 T_e$. Since we have dropped the requirement that $\int_S q \, dx dy = 0$, we obtain

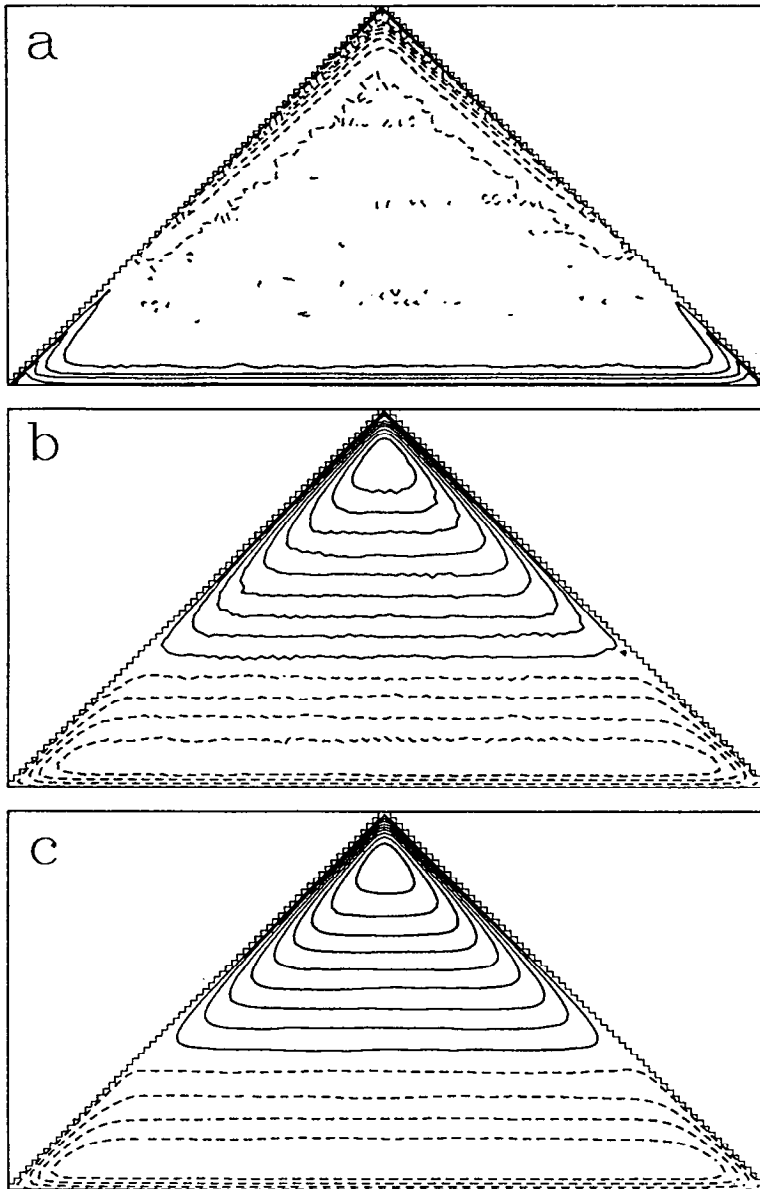


Figure 10. The smoothed time mean fields of (a) relative vorticity ζ ; (b) streamfunction ψ ; (c) and absolute vorticity q plus λ , for ISID2. Note the average is taken for only $10 T_e$.

a linear relationship in the form of (1.3) with $\lambda \neq 0$. Figure 11 shows the scatter plot of $(q + \lambda)$ vs. ψ . The plots are rather pleasing; clearly we have realized the Fofonoff solution in triangular domain.

In the spirit of the discussion in 3a, one can partly understand why ISID2 experiment succeeds in arriving a time mean Fofonoff state in a relatively short time,

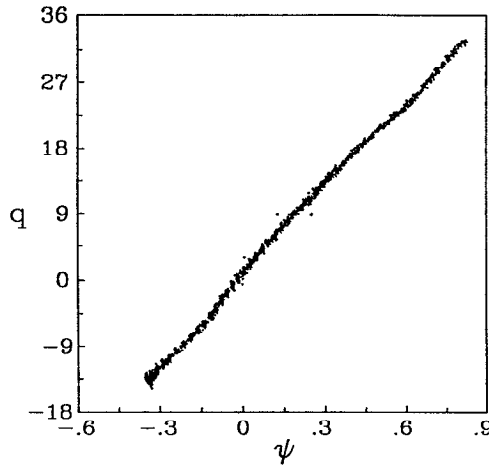


Figure 11. The scatter plot of $(q + \lambda)$ vs. ψ for experiment ISID2, for the smoothed time mean fields.

while ISID1 fails. However, the way in which the random initial field approaches the average Fofonoff state is less clear, for example in the way Rossby wave reflects on the side boundary of the triangular domain. Nevertheless, we believe that the time mean Fofonoff solution will finally be achieved even when the Rossby number is large, if the experiments are run for a long enough time.

4. Decaying experiments

Numerical simulations were performed to investigate the effect of various forms of viscosity and different boundary conditions, to test whether a small viscosity is indeed only a small perturbation, in some sense, to the inviscid dynamics, and how this depends on boundary conditions. If the perturbation is ‘small’, then a decaying simulation should presumably also be similar to a Fofonoff solution. (Note that the Fofonoff state is nonlinearly stable, by Arnold’s first theorem, and thus is little affected by small perturbations, if the dynamics themselves are unaltered.) If the perturbation has a large effect, then the results in more realistic ocean models may presumably also be sensitive to viscosity and boundary conditions.

Neither the nature of the eddy viscosities nor the correct type of the boundary conditions to use in ocean models are well understood. Here, harmonic and biharmonic viscosities along with free slip, no slip, and superslip boundary conditions are used, in part because they are the most widely used in the literature. (None of the three boundary conditions used is actually satisfied, or even approximately satisfied, by the Fofonoff solution.) We shall see that it is the form of the boundary condition, rather than that of viscosity, which largely determines the results of our decaying experiments. Therefore the following discussion is subgrouped by the boundary conditions used. Energy is not conserved in the simulations, so neither β -plane

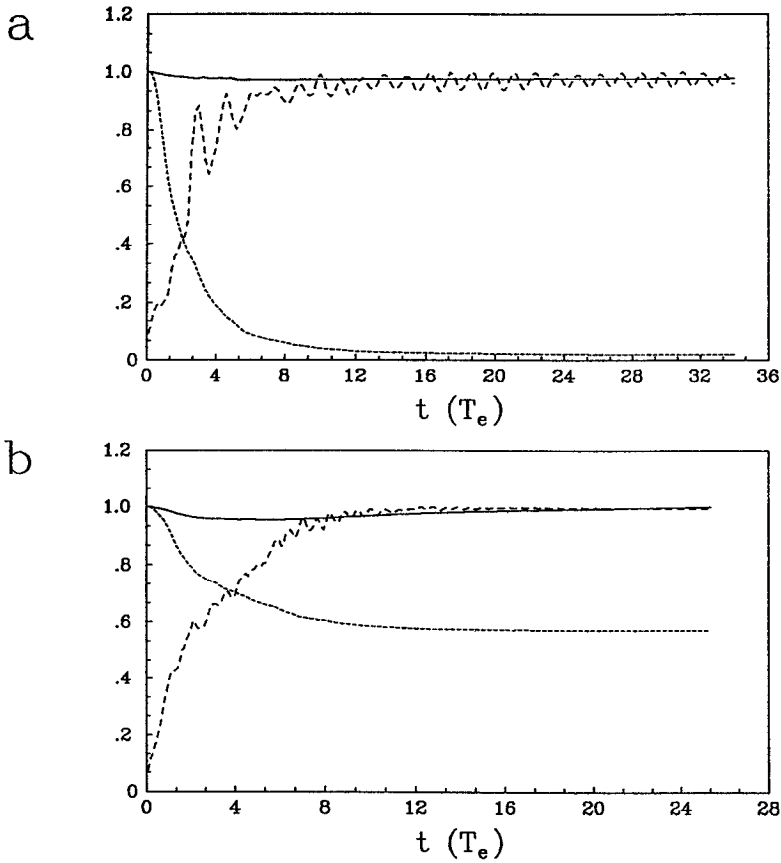


Figure 12. Time series of energy (solid curve), potential enstrophy (dashed curve) and anticorrelation function (long-dashed curve) for experiments (a) SS5 and (b) SS11.

Rossby number nor turnover time are precisely defined. However, energy may decay only slowly and we will use the initial values of R_o and T_e in the following discussion.

a. Experiments with super-slip boundary conditions. Superslip boundary condition conserves circulation, which, as the variational arguments suggest, might be very important for arriving at a linear relation between q and ψ . However, the change of the total energy or potential enstrophy does not have any definite sign: it depends on the field, the boundary values and certain line integrals. Figure 12 shows the typical time evolution of energy, potential enstrophy and anticorrelation function for initial β -plane Rossby number being 1.59×10^{-2} (a), and 3.18×10^{-3} (b) using superslip boundary condition. As long as the viscosity is small enough, demanding correspondingly high resolution, the energy typically varies only weakly and slowly, initially decreasing and then slowly increasing. Potential enstrophy decreases almost mono-

Table 2. The decaying experiments performed in rectangular domain with superslip boundary condition. The second column shows the initial β -plane Rossby numbers of the experiments, the initial energy is the same, but β is chosen differently. The fifth column shows the constants used in the harmonic viscosity $A_2 \nabla^2 \zeta$ or in the biharmonic viscosity $A_4 \nabla^4 \zeta$.

Run	R_0	Resolution	Viscosity	A_4 , or A_2
SS1	1.59×10^{-2}	64×64	harmonic	$1. \times 10^{-4}$
SS2	7.95×10^{-3}	64×64	harmonic	$1. \times 10^{-4}$
SS3	3.18×10^{-3}	64×64	harmonic	$1. \times 10^{-4}$
SS4	1.59×10^{-3}	64×64	harmonic	$1. \times 10^{-4}$
SS5	1.59×10^{-2}	128×128	biharmonic	$5. \times 10^{-7}$
SS6	1.59×10^{-2}	128×128	biharmonic	$1. \times 10^{-7}$
SS7	1.59×10^{-2}	128×128	biharmonic	$5. \times 10^{-8}$
SS8	3.18×10^{-3}	128×128	biharmonic	$5. \times 10^{-7}$
SS9	3.18×10^{-3}	128×128	biharmonic	$1. \times 10^{-7}$
SS10	3.18×10^{-3}	128×128	biharmonic	$5. \times 10^{-8}$
SS11	1.59×10^{-3}	128×128	biharmonic	$5. \times 10^{-7}$
SS12	1.59×10^{-3}	128×128	biharmonic	$1. \times 10^{-7}$
SS13	1.59×10^{-3}	128×128	biharmonic	$5. \times 10^{-8}$
SS14	1.59×10^{-2}	128×128	biharmonic	$5. \times 10^{-7}$
SS15	1.59×10^{-2}	128×128	biharmonic	$1. \times 10^{-7}$
SS16	1.59×10^{-2}	128×128	biharmonic	$5. \times 10^{-8}$
SS17	3.18×10^{-3}	128×128	biharmonic	$5. \times 10^{-7}$
SS18	3.18×10^{-3}	128×128	biharmonic	$1. \times 10^{-7}$
SS19	3.18×10^{-3}	128×128	biharmonic	$5. \times 10^{-8}$
SS20	1.59×10^{-3}	128×128	biharmonic	$5. \times 10^{-7}$
SS21	1.59×10^{-3}	128×128	biharmonic	$1. \times 10^{-7}$
SS22	1.59×10^{-3}	128×128	biharmonic	$5. \times 10^{-8}$
SS23	1.59×10^{-2}	256×256	biharmonic	$1. \times 10^{-8}$
SS24	3.18×10^{-3}	256×256	biharmonic	$1. \times 10^{-8}$
SS25	1.59×10^{-3}	256×256	biharmonic	$1. \times 10^{-8}$
SS26	1.59×10^{-2}	256×256	biharmonic	$1. \times 10^{-8}$
SS27	3.18×10^{-3}	256×256	biharmonic	$1. \times 10^{-8}$
SS28	1.59×10^{-3}	256×256	biharmonic	$1. \times 10^{-8}$

tonically to certain value, then stays almost constant. The value of this constant Q_2 depends on the initial β -plane Rossby number: if R_0 is big, e.g. 1.59×10^{-2} , $Q_2 \sim 0$. On the other hand, if R_0 is small, Q_2 will not be decreased to zero, although the decrease of Q_2 is still much larger compared with the decrease of E . Table 2 summarizes the experiments conducted with harmonic or biharmonic viscosity using the super-slip condition.

SS1 to SS4 are a group of experiments with harmonic viscosity and a relatively small Reynolds number— $R_e \sim 4 \times 10^4$. The difference between them is in the β -plane Rossby numbers used (Table 2). The anticorrelation functions experience a similar time variation as in the inviscid experiments. Depending on the Rossby number R_0 , the initially random fields in these four experiments spin up in less than

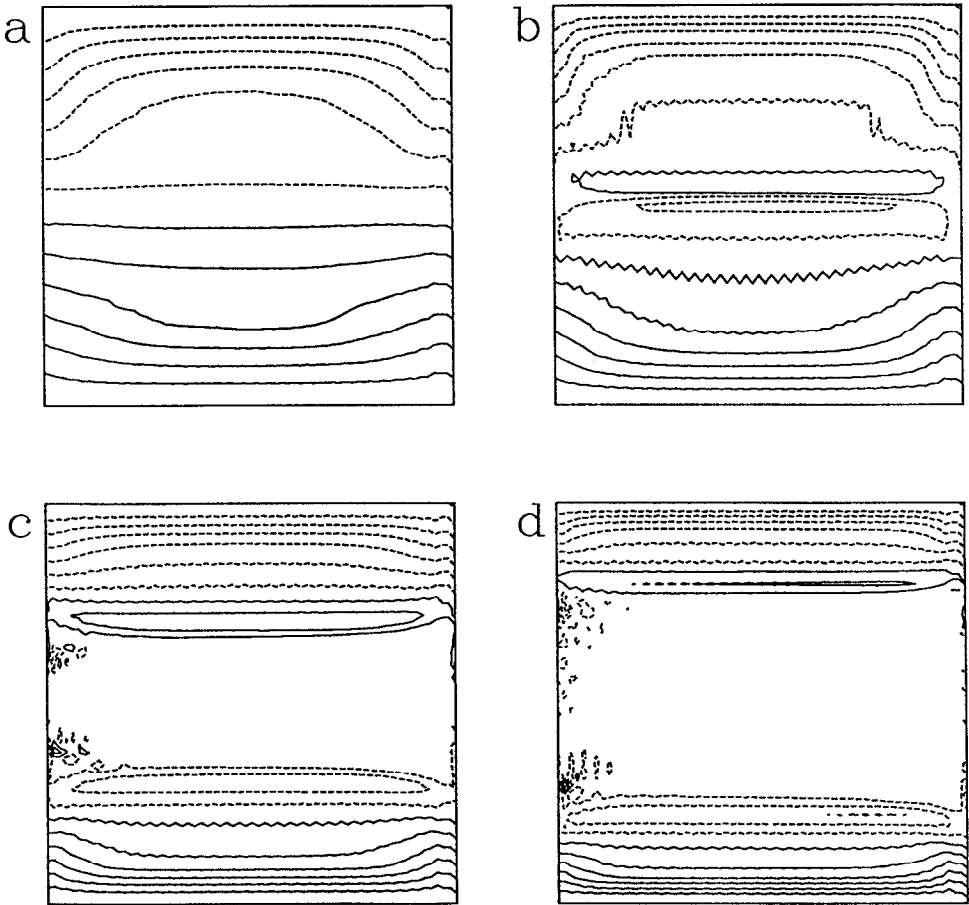


Figure 13. Time mean relative vorticity ζ fields in experiments (a) SS1, (b) SS2, (c) SS3, and (d) SS4.

about $10 T_e$ to quasi-equilibrium states. Figures 13, 14, 15 show the time mean ζ , ψ , and q for the four experiments, and Figure 16 shows the scatter plots of $q - \psi$. From these figures, one sees: (a) antisymmetry of the northern and southern basin, which is stronger for bigger Rossby number case; (b) homogenization of the q fields, which is more obvious and more important in smaller R_o case. Similar results are obtained with biharmonic viscosity in a corresponding parameter range.

It is interesting to note that the simulations ISRD2 and SS4, which share the same β -plane Rossby number, experience a similar early evolution to that depicted in Figure 4. Ultimately, differences arise after two gyres form near the northern and southern walls: in the inviscid simulation, the two gyres grow and eventually fill up the whole basin; whereas in the decaying case, the fields settle down to states in which the ψ is concentrated along the northern and southern boundary walls, and the

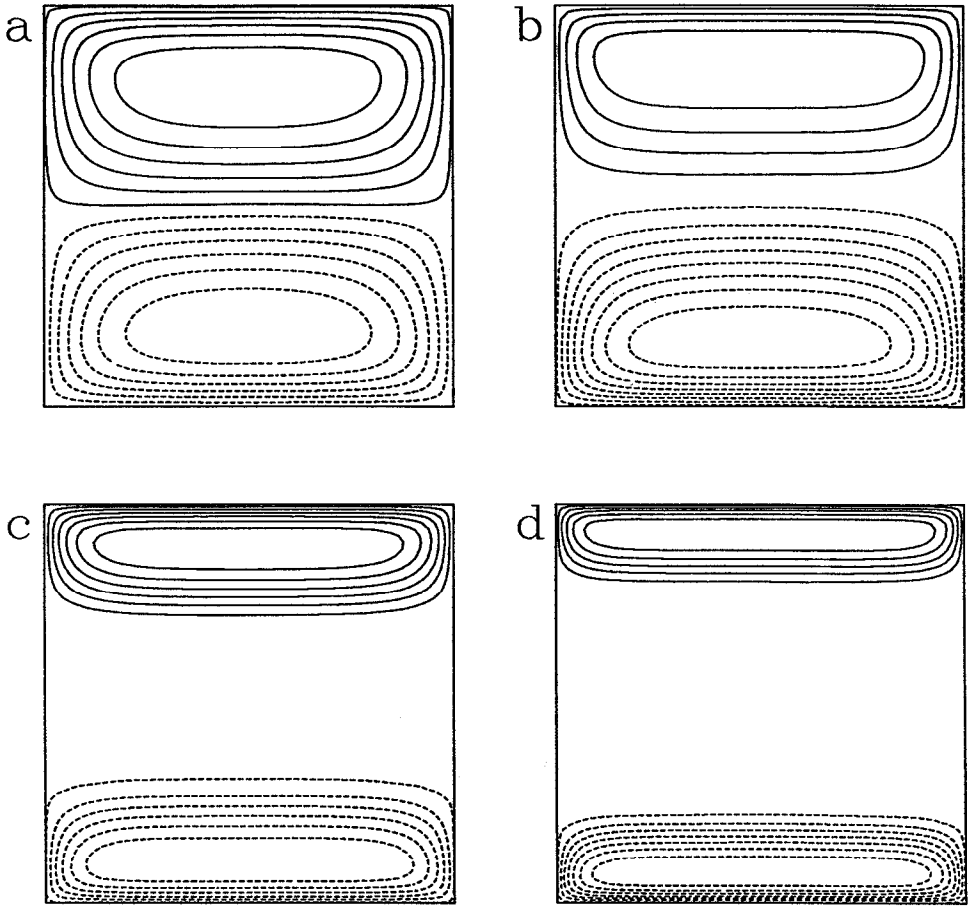


Figure 14. Time mean streamfunction ψ fields in experiments (a) SS1, (b) SS2, (c) SS3, and (d) SS4.

small scale structures are then basically ‘mopped up.’ We discern two competing mechanisms: one is the nonlinear interaction which tends to drive the fields to a time mean Fofonoff solution; the other is the viscous dissipation which seems to drive the fields toward homogenization. With small Reynolds number, especially when the Rossby number is also small, dissipation becomes dominant soon after a few eddy turnover times.

Cessi *et al.* (1987) studied the relationship between the value of potential vorticity inside the gyre and the gyre structure. In the northern homogenized gyre they predict:

$$q_I = \beta \frac{y_n + 2y_s}{3}, \quad (4.1)$$

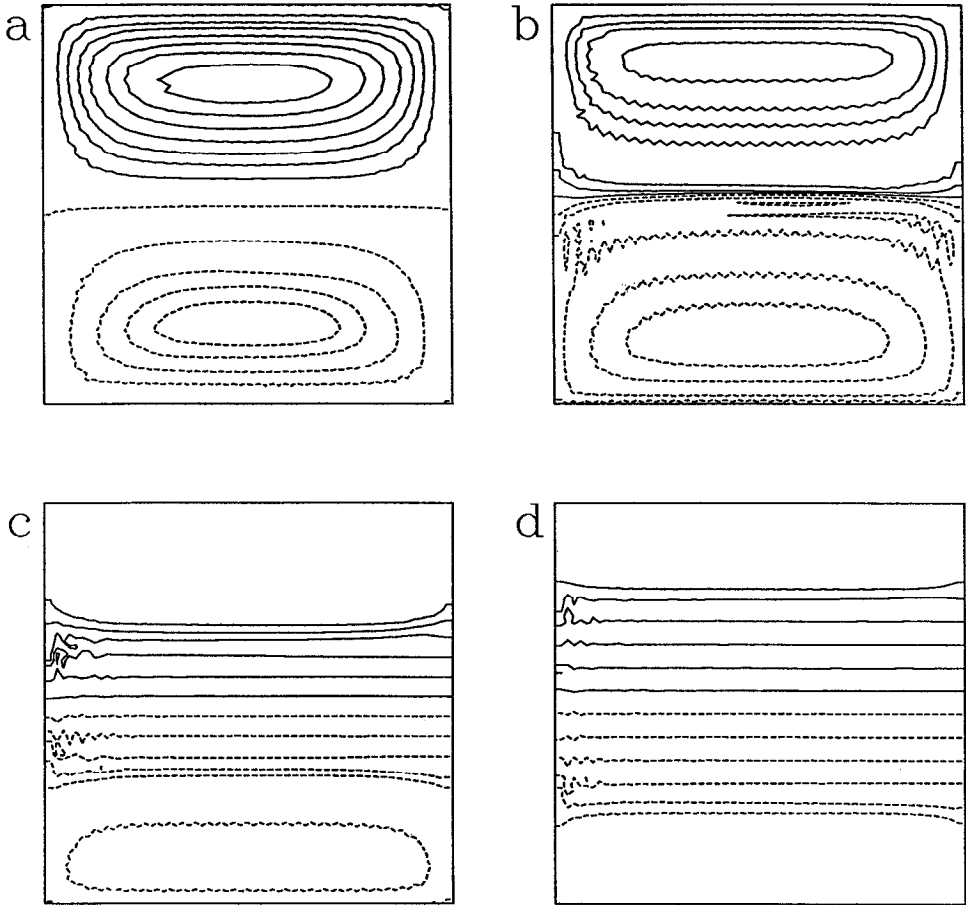


Figure 15. Tim mean absolute vorticity q fields in experiments (a) SS1, (b) SS2, (c) SS3, and (d) SS4.

where q_I is the average value of the potential vorticity inside the northern gyre, y_s is the southern boundary of the gyre and y_n is just inside the frictional boundary layer at the northern wall. The estimates of q_I for SS3 and SS4 from (4.1) give 9.9 and 23.5 respectively, in good agreement with the real values 8.9 and 22.

To further pursue the influence of viscosity, we did some parameter-sensitivity experiments with biharmonic viscosity at higher resolution. Experiments SS5 to SS13 are done with resolution 128×128 , three values of R_o are used, namely 1.59×10^{-2} , 7.95×10^{-3} and 3.18×10^{-3} . Three values of the viscous constant A_4 are used, namely $5. \times 10^{-7}$, $1. \times 10^{-7}$ and $5. \times 10^{-8}$, which make the “pseudo-Reynolds number” $R_e = U_{rms} L^3 / A_4$ equals 3.1×10^8 , 1.55×10^9 , 3.1×10^9 respectively. These larger values are of course still small compared to the Reynolds numbers in real ocean. Figure 17 summarizes the results as scatter-plots of $q - \psi$. Figure 18 shows the results from

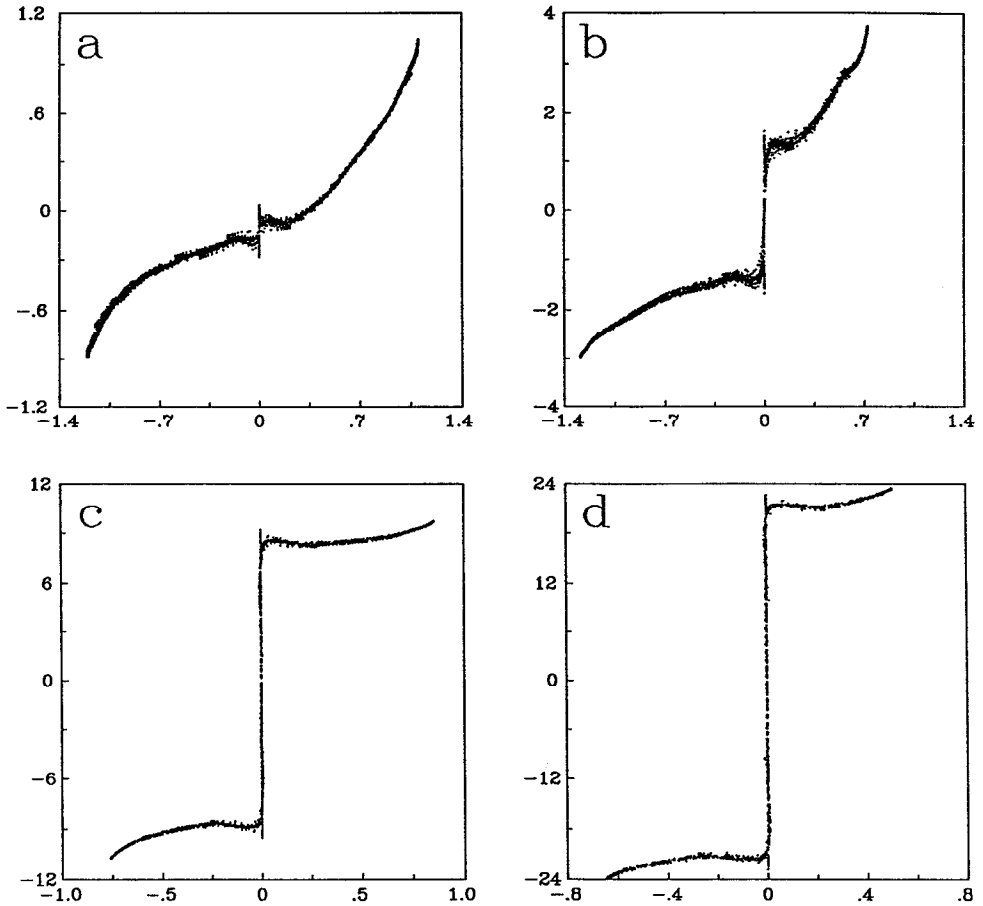


Figure 16. Scatter-plots of $q - \psi$ in experiments (a) SS1, (b) SS2, (c) SS3, and (d) SS4. q is plotted against ψ in all panels.

experiments SS14 to SS22; each experiment in Figure 18 shares the same parameters as its counterpart in Figure 17 except that initial condition used in the experiments shown in Figure 17 is different from the one used in those shown in Figure 18: the initial energy value and energy spectra are the same, but the phases differ.

Asymmetry of the northern and southern basin, and homogenization in the fields occurs to some extent in all these 9 pairs of experiments. In the inviscid experiments, we commented that the statistical mechanical equilibrium was more easily achieved for small Rossby number. However, the differences there arose in terms of timescales and scatter in the plot, not in the functional form achieved, which was close to linear in all cases. Here, as Rossby number decreases, homogenization of the gyres is more easily achieved. However as Reynolds number increases, the fields tend to become less homogenized; put another way, the larger the Rossby numbers (weaker β) the

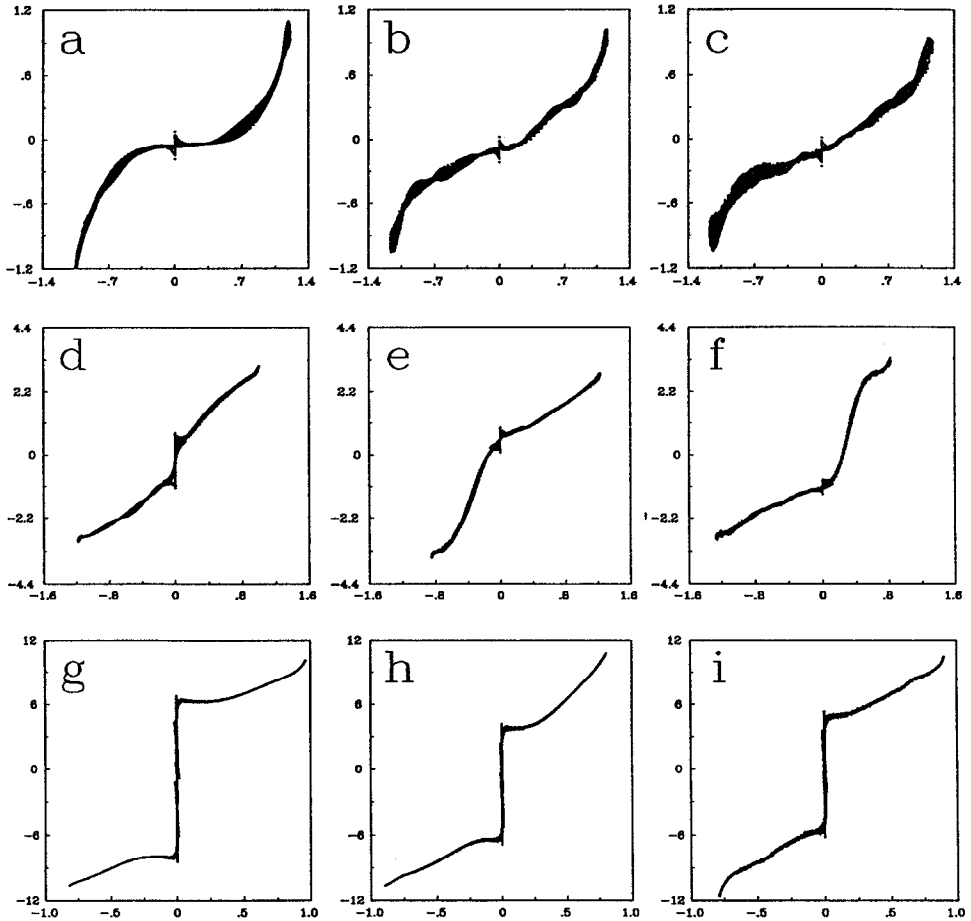


Figure 17. Scatter-plots of $q - \psi$ in 128×128 experiments (a) SS5, (b) SS6, (c) SS7, (d) SS8, (e) SS9, (f) SS10, (g) SS11, (h) SS12 and (i) SS13. The experiments share the same biharmonic viscosity with superslip boundary conditions. Those in the same line share the same Rossby numbers, which, from top to bottom, are 1.59×10^{-2} , 7.95×10^{-3} and 3.18×10^{-3} . Those in the same column share the same Reynolds numbers, which, from left to right, are 3.1×10^8 , 1.55×10^9 and 3.1×10^9 .

more linear the $q - \psi$ relationship. Also, as Reynolds number increases (at constant Rossby number) the homogenized zones decrease in extent. Comparison between Figures 17 and 18 shows that, although the similarity is more prominent, dependence of the initial condition exists, especially in the medium Rossby number cases.

Figure 19 shows the scatter plots of a group of higher resolution experiments SS23–SS28; the Reynolds numbers are 1.55×10^{10} for all. The experiments in the same lines have the same Rossby number, the two columns representing different initial conditions. Although the detailed results depend on the initial conditions, it is

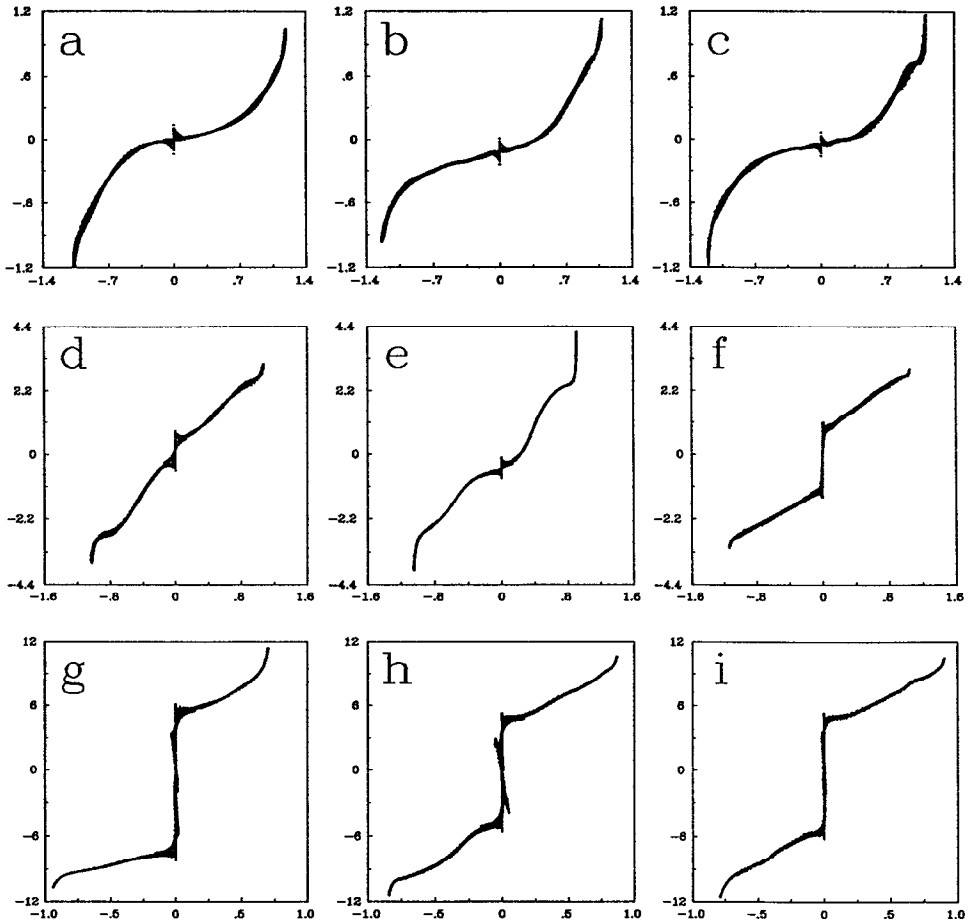


Figure 18. Scatter-plots of $q - \psi$ in 128×128 experiments (a) SS14, (b) SS15, (c) SS16, (d) SS17, (e) SS18, (f) SS19, (g) SS20, (h) SS21 and (i) SS22. Each plot has the same parameters as its counterpart in Figure 17, except that the experiments started from different initial random flow.

clear that the homogenization zones decrease in extent as Rossby number increases. However homogenization remains, and the $q - \psi$ relationship fails to become linear, even as the Reynolds number increases (in the parameter regime available to us).

b. Experiments with free-slip boundary conditions. Experiments with free-slip boundary conditions are essentially freely decaying turbulence in a closed domain. The viscosity should of course be chosen to be small so that the spin up time to a quasi-equilibrium will be much smaller than an energy decay timescale. Since the magnitude of viscosity is determined by the requirement to resolve the frictional boundary layer, this entails correspondingly high resolution.

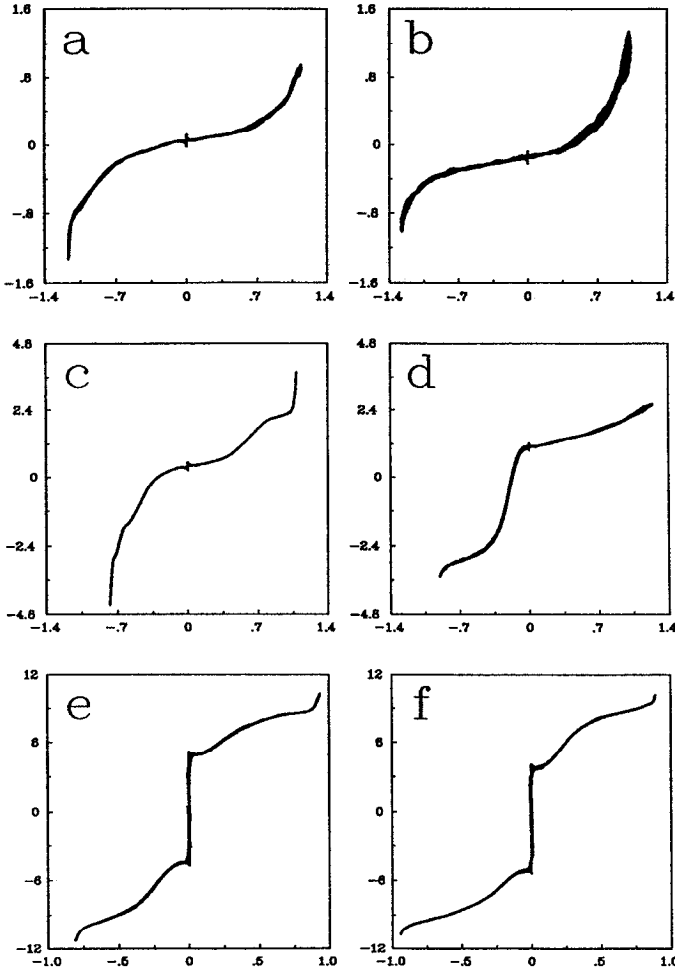


Figure 19. Scatter-plots of $q - \psi$ in 256×256 experiments (a) SS23, (b) SS26, (c) SS24, (d) SS27, (e) SS25, (f) SS28. The experiments share the same biharmonic viscosity with superslip boundary conditions. Those in the same line share the same Rossby numbers, which, from top to bottom, are 1.59×10^{-2} , 7.95×10^{-3} and 3.18×10^{-3} . Those in the same column share the same initial condition. The Reynolds number is the same for all, which is 1.55×10^{10} .

The thickness of the frictional boundary layer can be estimated by balancing the β -term and viscosity. With biharmonic viscosity, this gives a thickness $\sim (A_4/\beta)^{1/5}$, which should be larger than the grid size. Table 3 summarizes some of the experiments we did with the free slip boundary condition; for all of them, only biharmonic viscosity is used.

Table 3. The decaying experiments performed in rectangular domain with free slip boundary condition. The second column shows the initial β -plane Rossby numbers of the experiments, the initial energy is the same, but β is chosen differently. The fifth column shows the constants in the biharmonic viscosity $A_4 \nabla^4 \zeta$.

Run	R_0	Resolution	Viscosity	A_4
FS1	1.59×10^{-2}	128×128	biharmonic	$5. \times 10^{-7}$
FS2	7.95×10^{-3}	128×128	biharmonic	$5. \times 10^{-7}$
FS3	1.59×10^{-2}	256×256	biharmonic	$5. \times 10^{-7}$

A simple calculation shows that now the energy will decrease monotonically according to

$$\frac{dE}{dt} = -A_4 \int_S (\nabla \zeta)^2 dx dy, \quad (4.2)$$

but this rate may be slow compared to other dynamical timescales. The circulation is not conserved, unlike in the superslip experiments. Time-series of energy, potential enstrophy and anticorrelation function in a typical large Rossby number (1.59×10^{-2}) experiment are shown in Figure 20. We see energy decreases slowly, showing no obvious indication of settling down; potential enstrophy decreases to a minimum, and then plateaus to stay more-or-less constant; in this case, almost zero. The anticorrelation function increases to some maximum, and then oscillates around a slowly decaying mean value. It is clear from Figure 20 that the fields are constantly decaying, no equilibrium or quasi-equilibrium state has been strictly achieved.

Experiments FS1 and FS3 share the parameters: R_0 , which is the same as that of ISRD1, and R_e , but FS3 has higher resolution, hence is better resolving the frictional boundary layers. The qualitative properties of these two experiments are the same. Figure 21 show the time mean fields and scatter plots for FS3. The streamfunction resembles that of the Fofonoff state. However, although the potential vorticity field does have two gyre structure, it is affected by a strong boundary layer effect and is not parallel to the streamlines. This boundary layer effect is also prominent in Figure 21d, the scatter-plot in the whole basin. Figure 21e shows the scatter-plot of $q - \psi$ of the interior points, excluding the first 12 layers closest to the walls (i.e. excluding those points close to $\psi = 0$ in the scatter-plot). It shows a functional relation between q and ψ , which is similar to Figures 17a and 18a. It is clear while the dissipation contributes largely the departure of the scatter-plot from linearity, as in the super-slip experiments, the frictional boundary layer further exacerbates the departure in free-slip experiments. Note that the boundary potential vorticity is tied to the value βy , and is not free to respond to the dynamics.

The time mean fields and scatter plots for FS2 are shown in Figure 22. The streamfunction field is not like that of Fofonoff solution, or that of the well

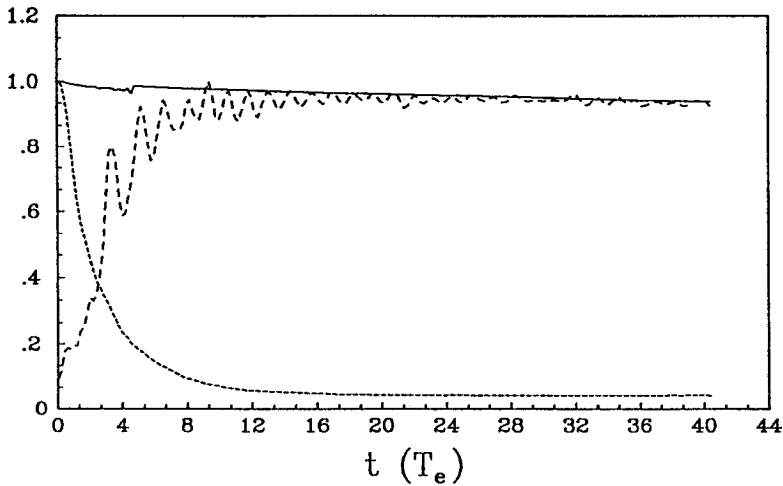


Figure 20. Time series of energy (solid curve), potential enstrophy (dashed curve) and anticorrelation function (long-dashed curve) for experiments FS1.

homogenized field in Section 4a. Although the fields are somewhat noisy, homogenization is clearly apparent. The scatter-plots indicate that the main departure from linearity is again due to both the contribution from frictional boundary layer and homogenization.

c. Experiments with no-slip boundary conditions. With no slip boundary conditions, we performed experiments only with harmonic viscosity. Again, one has to resolve the frictional boundary layer to avoid numerical problems, and we have done only limited parameter-sensitivity experiments.

In all of our no-slip experiments, energy and potential enstrophy quickly decay. The time mean fields and the corresponding scatter-plots show little but the turbulent fields with almost zero magnitudes. No equilibrium or quasi-equilibrium state is ever observed.

One important feature of no-slip boundary condition is that the anticorrelation function is always zero. That is,

$$\begin{aligned}
 C &\equiv - \int_S \zeta \beta y dx dy \\
 &= -\beta \left(\int y \frac{\partial \psi}{\partial n} ds - \int \frac{\partial \psi}{\partial y} dx dy \right) = 0.
 \end{aligned}
 \tag{4.3}$$

Zero anticorrelation function means that any correlation of ζ field with latitude is forbidden. To generalize further, one can prove that for any function f and g ,

$$\int_S \zeta f(y) dx dy = \int_S \zeta g(x) dx dy = 0.
 \tag{4.4}$$

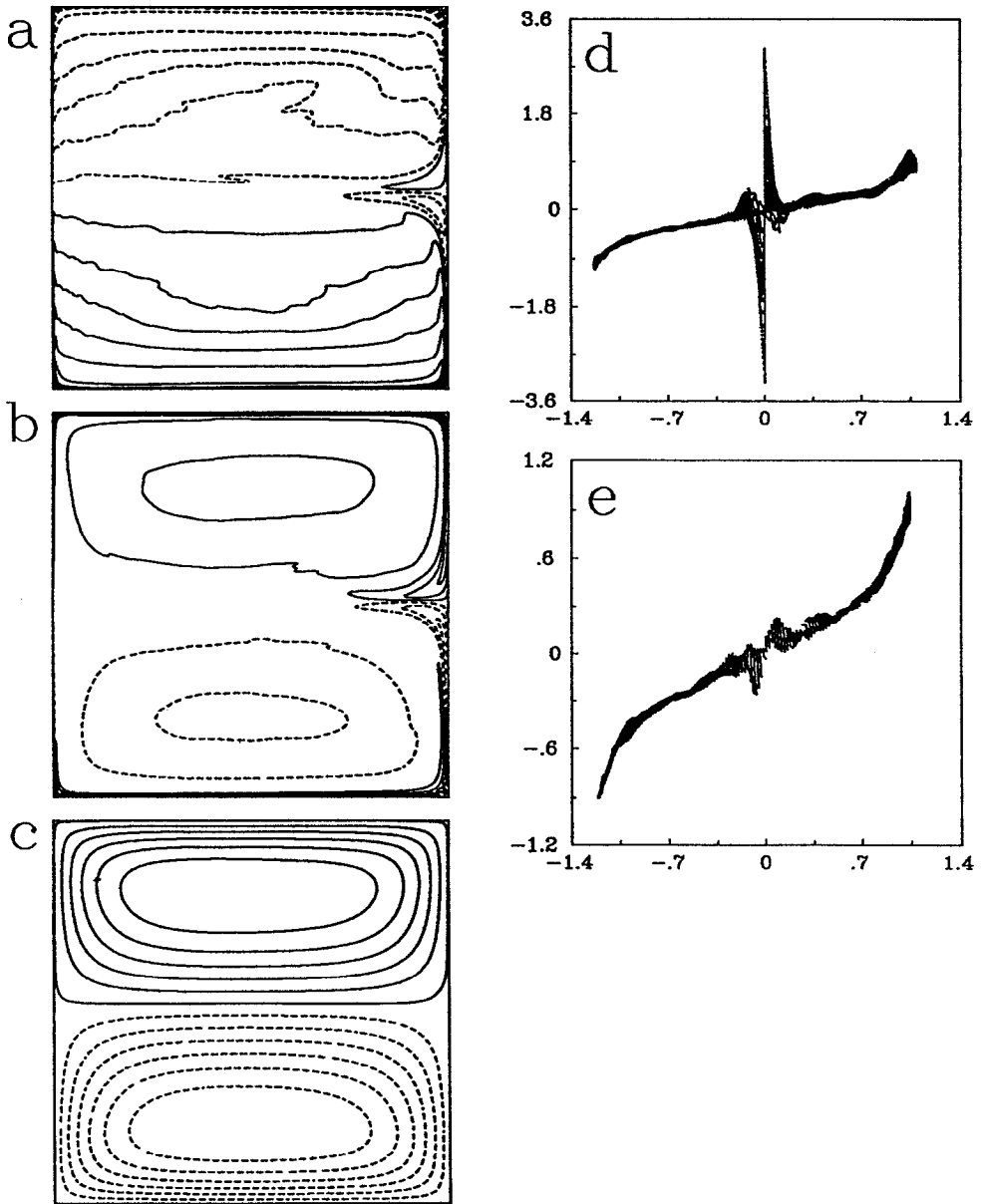


Figure 21. Counter plots of the time mean fields, and scatter-plots of $q - \psi$ of experiment FS3. Average is done for $10 T_e$ after the first $20 T_e$, shown here (a) is relative vorticity ζ , (b) absolute vorticity q , (c) streamfunction ψ , (d) scatter-plot of $q - \psi$ for the entire domain, and (e) scatter-plot of the interior points, excluding the first 12 layers closest to the walls.

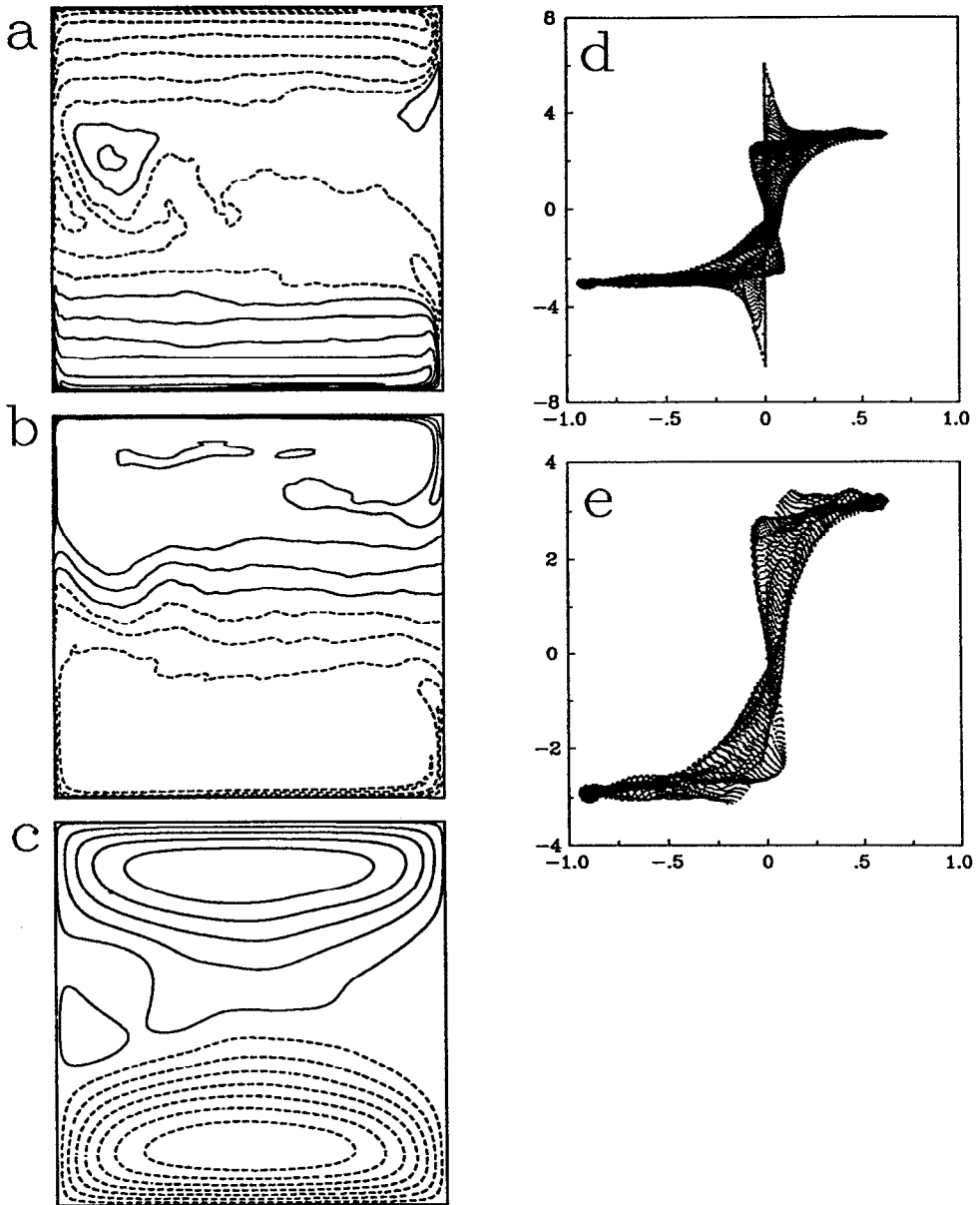


Figure 22. Contour plots of the time mean fields, and scatter-plot of $q - \psi$ of experiment FS2. The time average is done for $10 T_e$ after the first $20 T_e$. (a) relative vorticity ζ ; (b) absolute vorticity q ; (c) streamfunction ψ ; (d) scatter-plot for the entire domain; (e) the scatter-plot of $q - \psi$ of the interior points, excluding the first 12 layers closest to the walls.

This means that the equilibrium or quasi-equilibrium, were one to exist, would be quite different from a Fofonoff state.

5. Experiments with topography

In this section we briefly report on inviscid experiments of flow over topography. We have done experiments with topography designed to (roughly) imitate continental slopes, seamounts and ridges. As well as the theoretical interest in understanding such simple systems, the generation of mean flows by eddy-topographic interaction is of direct oceanographic interest (see also Vallis and Maltrud, 1993; Holloway, 1992).

The flow on the β -plane discussed above is in fact a special case of flow over topography, with $h(x, y) = \beta y$. In the general case the theory predicts a time-mean field given by (1.3) with βy being replaced by a general topographical function $h(x, y)$:

$$\nabla^2 \langle \psi \rangle + h(x, y) = \mu \langle \psi \rangle + \lambda. \quad (5.1)$$

For the three different topographical functions $h(x, y)$ shown in Figure 23 we solve Eq. (5.1). The solutions are plotted in Figure 24. The parameters are so chosen that $\int_S \zeta \, dxdy = 0$. These are the solutions obtained from the linear relationship between $q = \nabla^2 \psi + h(x, y)$ and ψ and expected by the equilibrium theory; it is therefore appropriate to call this whole family of solutions the ‘topographical Fofonoff solution’.

Our experiments have topography but no β -effect. We may vary the magnitude of the topography by varying the parameter b in the expression $h(x, y) = bh_b(x, y)$. By increasing (decreasing) the parameter b , we can increase (decrease) the steepness of the topography. Experiments IT1, IT2 and IT3 use the topographical function shown in Figures 23a, b, and c respectively, and hence imitate a system with continental slope, seamounts and a ridge, respectively.

The timescales for arriving at the expected topographical Fofonoff states are longer than in β -plane case, because Rossby waves are unable to propagate over the entire domain, an effect especially noticeable in the case of the isolated seamounts. In the cases of the continental slope (IT1, $b = 16$) and the ridge (IT3, $b = 8$) equilibrated time mean fields form after about $200 T_e$. However, a case with two isolated seamounts (IT2, $b = 16$) keeps evolving even after four or five hundred eddy turnover times, with only a very slow tendency to a single linear $q - \psi$ relationship.

In the cases of IT1 and IT2, some quasi-equilibrium states are experienced before the time mean Fofonoff solution is achieved. Figure 25a and b show the scatter-plots of the $q - \psi$ fields of IT1 at two stages of evolution. Initially, different parts of the domain yield differing relationships between q and ψ . Eventually, a single linear relationship between q and ψ is reached over the entire domain, as would be demanded by ergodicity, but only after a very long time—typically of order one hundred eddy turnover times, compared with $\sim 10 T_e$ in ISRD1 and ISRD2. Similarly, IT2 (Figs. 25c and d), the experiments over two seamounts show that,

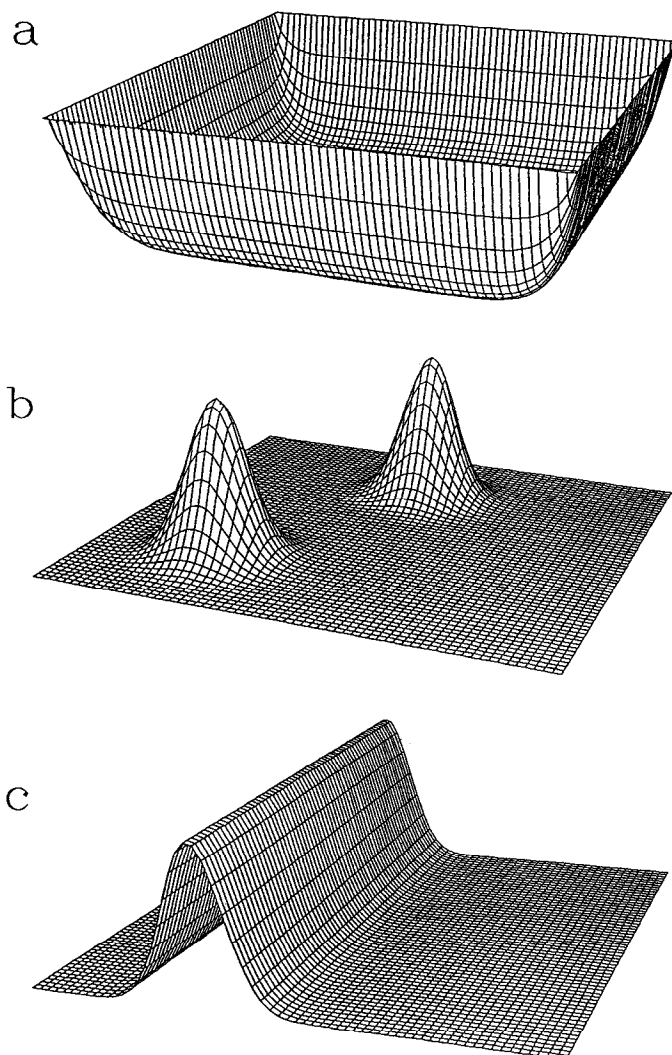


Figure 23. Three bottom topographies used in experiments discussed in the Section 5. They are for IT1, IT2, IT3 respectively.

initially, local quasi-equilibrium states are achieved over each seamount, each characterized by a different functional relationship between q and ψ . There is a slow tendency toward a global equilibrium state, indicated by the convergence of the scatter plots over the two seamounts. The time to reach the single equilibrium is very long, greater than several hundred turn-over times, although it seems clear the system is in fact evolving toward it. In these experiments Rossby waves only propagate where there is changing topography, namely over the mounts themselves;

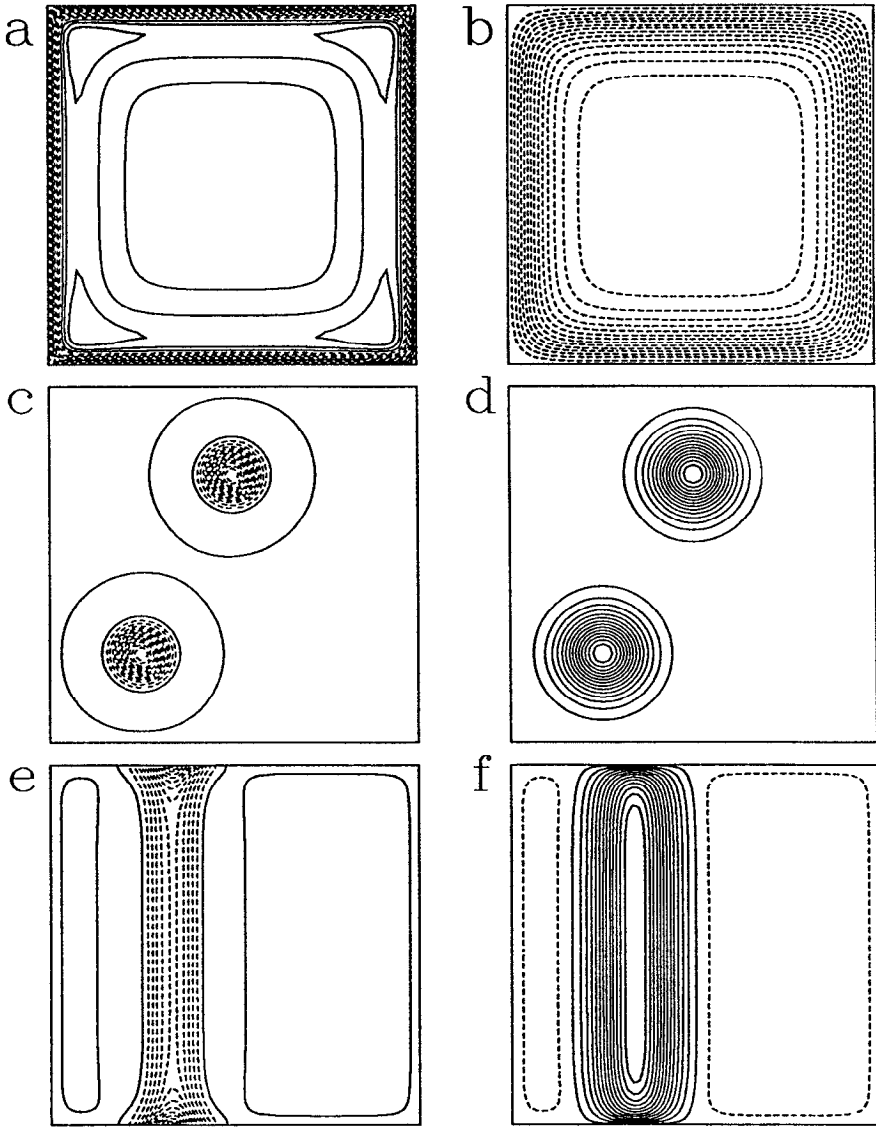


Figure 24. Contour plots of relative vorticity (a), streamfunction (b) of the exact topographical Fofonoff solution when the topography shown in Figure 23a is used. (c), (d) are those corresponding to topography in Figure 23b; (e), (f) are those corresponding to topography in Figure 23c. Parameters are so chosen that the circulation is zero in all cases.

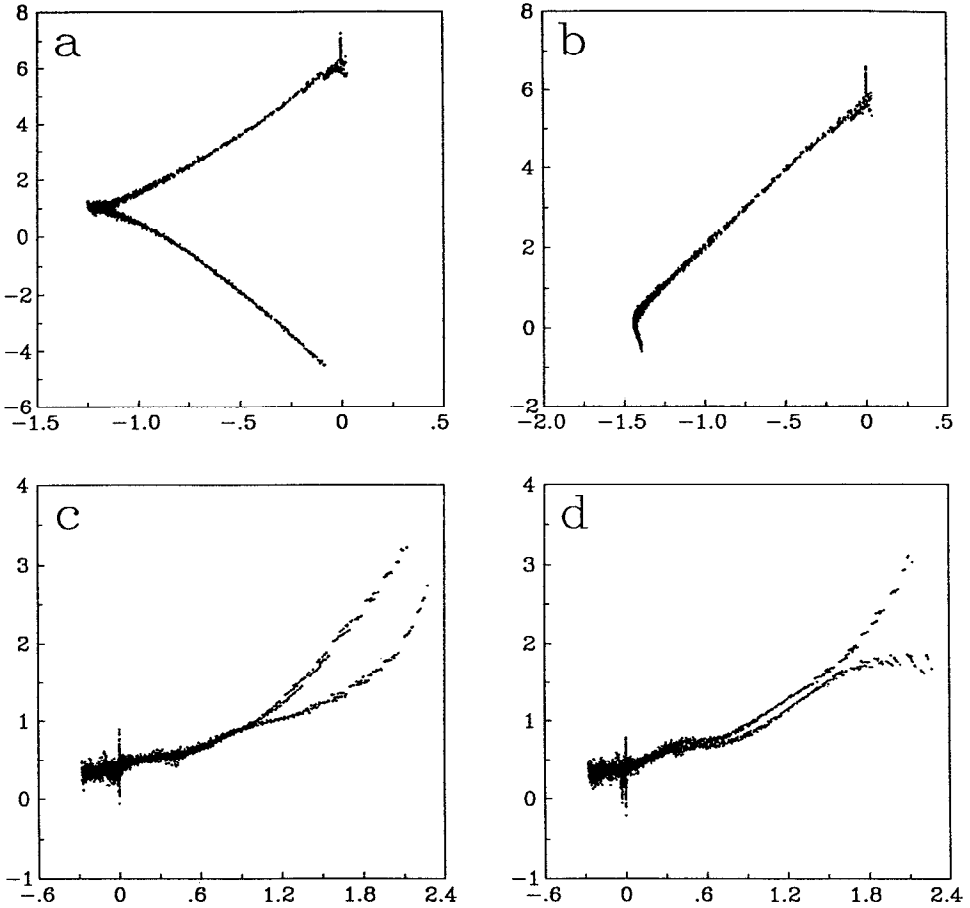


Figure 25. (a) and (b) are the $q - \psi$ scatter-plots of experiment IT1, and (c) and (d) are of IT2, all for the time mean fields. (a) IT1, time averaged over time between $20 T_e$ to $70 T_e$; (b) IT1, averaged between $70 T_e$ to $170 T_e$; (c) IT2, averaged over time between $65 T_e$ to $165 T_e$; (d) IT2, averaged between $350 T_e$ to $450 T_e$.

thus communication between the mounts is very inefficient and it is difficult for the system to evolve into a maximum entropy state. Consistent with this, experiments with small topography (say, $b = 2$) take even longer to evolve to a linear relationship.

6. Discussion

The relevance of statistical models to real, viscous, fluids depends in part on how much their behavior is influenced by the presence of small viscous terms. Of course, even if such small terms have a disproportionate effect, the approach or the tendency to a statistical mechanical equilibrium may yet manifest itself in a forced-viscous

model and thereby reveal important tendencies. The final equilibrium of such a model may then be a balance between the competing influences of a tendency toward statistical mechanical equilibrium and non-adiabatic effects. In the limit of infinite resolution, statistical mechanics does in fact make a prediction of the time-mean state in a two-dimensional closed domain, namely that of a Fofonoff flow. At finite resolution, the form of the solution is the same, although the numerical parameters differ somewhat.

This paper addressed, first, the issue of whether a time mean Fofonoff solution actually obtains in an inviscid simulation. Without ascertaining whether such an inviscid equilibrium exists we have no basis from which to determine its relevance to the viscous problem. A strict test of this must eschew the use of viscosity, or a subgrid-scale parameterization, and the numerical experiments reported here in Section 3 are truly inviscid. Except for the one in triangular domain with a large Rossby number, all the experiments exhibit the time mean flow predicted by ergodicity; that is, all achieve the Fofonoff solution. The results show that a Fofonoff solution is more easily achieved in the small Rossby number case. This is because the Rossby waves are much more effective than pure turbulence in accelerating the convergence to a statistical mechanical equilibrium. However, in general, all our inviscid simulations eventually produce a mean field very close to the Fofonoff solution. The generalization to a triangular domain implies that the domain shape is not an important factor. To conclude, nonlinear interactions do drive the inviscid system, from an arbitrary random state, to a time mean Fofonoff solution, as a result of entropy maximization. There is no clear evidence that any of the systems are not ergodic. All these points are further qualitatively confirmed by the experiments over topography.

When viscosity, of any form, is added, the assumptions of equilibrium statistical mechanics are strictly violated, because phase space volumes are no longer preserved and the assumption of 'equal *a priori* probabilities' can no longer be made. In general, this precludes the achievement of a maximum entropy state, although it does not necessarily prevent the emergence of a Fofonoff state, if by that is meant a linear relationship between streamfunction and potential vorticity, since this is also predicted by minimum enstrophy arguments. However, such a minimum enstrophy state is rarely reached: a linear relationship between q and ψ does not arise both because of homogenization in the q field, and because of frictional boundary layers. The effects of the latter can be minimized by choice of boundary conditions, but it appears that the former cannot be avoided.

The importance of the frictional boundary layers is determined by the boundary condition; the detailed form of viscosity used appears less important. (This is consistent with the results of Cummins, 1992.) In an extreme case with no-slip boundary, no equilibrium state at all is realized in our experiments. When the use of

free-slip boundary condition is made, circulation and potential enstrophy will be dissipated, with energy dissipation being slow compared to that of potential enstrophy. Homogenization contributes to the departure from Fofonoff solution inside the basin, while frictional boundary layer exacerbates the departure near the boundaries. Experiments with superslip boundary conditions come closest of the dissipative runs to Fofonoff flows with a linear $q - \psi$ relationship. Such a boundary condition conserves circulation, and, at high Reynolds number, approximately conserves energy. At the same Reynolds number, experiments with a large Rossby number dissipate proportionately less energy than potential enstrophy, and a state close to a Fofonoff solution is obtained. At smaller Rossby number, however, potential vorticity q is readily homogenized and a linear $q - \psi$ relationship does not emerge, although the time mean state still displays a functional relationship between q and ψ . With medium to small Rossby numbers, our experiments fail to give strong evidence that the steady state will converge to a Fofonoff solution as Reynolds number becomes larger and larger; rather there seem to exist quasi-equilibrium states which depend on the initial conditions. However, because of the limitations of our finite numerical methods, we cannot definitively preclude the possibility that in the *limit* of infinite Reynolds number (requiring a solution of the continuous equations) a linear relationship between q and ψ will be achieved at all β -Rossby numbers.

In summary, the inviscid simulations do generally evolve into a Fofonoff state. However, the decaying simulations do not always evolve into a minimum potential enstrophy state, especially for small β -Rossby number, even at fairly high resolution. It may be profitable to think of the actual time evolution of a system as determined by a ‘competition’ between the tendency of free evolution to produce an equilibrium state, and the potentially disruptive effects of forcing, dissipation and boundary conditions. Whether these effects can be quantitatively and usefully included in a non-equilibrium statistical theory remains to be seen.

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APPENDIX

Boundary layer solutions in a triangular domain

Here we derive two boundary layer solutions in a triangular domain, for the Fofonoff flow and the Stommel model. The method used is similar to that described in Munk and Carrier (1950) for the Munk problem (i.e. harmonic viscosity) in a triangular domain.

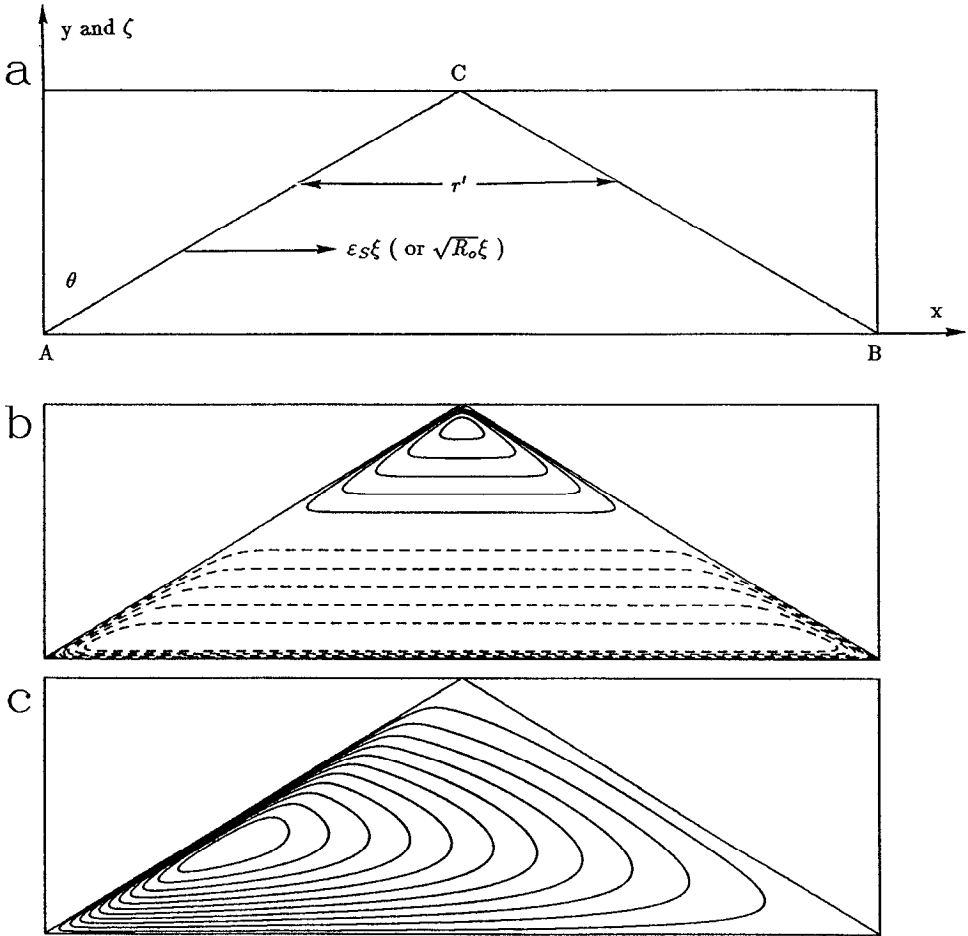


Figure 26. (a) The domain and the parameter used in the boundary layer calculation. (b) The streamfunction contour plot of the boundary layer Fofonoff solution (A8) with $R_o = 4. \times 10^{-4}$, $y_o = 0.5$ and $a = 3.25$. (c) The streamfunction contour plot of the boundary layer Stommel solution (A18) with $\epsilon_S = 0.01$, and $a = 3.25$.

1. Fofonoff solution

Consider the linear relation between q and ψ in nondimensionalized form:

$$R_o \nabla^2 \psi + y - y_o = \psi, \quad (\text{A1})$$

where the β -plane Rossby number R_o is small, and we set the speed of westward zonal flow to be unity. The triangular domain is embedded in a $a \times 1$ rectangle, the angle between the western boundary and vertical direction is ϑ (see Fig. 26a).

Away from the boundary, we can write the solution of (A1) immediately:

$$\psi^I = y - y_o. \quad (\text{A2})$$

Now we do the coordinate transformation:

$$\xi = \frac{x}{\sqrt{R_o}} - \frac{\zeta}{\sqrt{R_o}} \tan \vartheta, \quad (\text{A3a})$$

$$\zeta = y, \quad (\text{A3b})$$

so that $\sqrt{R_o}\xi$ gives the distance from the western boundary. Now the western and eastern boundary are given by $\xi = 0$ and $\xi = r' / \sqrt{R_o}$, where r' is the width given by $r' = a(1 - \zeta)$. Near $y = 0$,

$$\psi = \psi^I + \psi^S,$$

where

$$R_o \frac{\partial^2 \psi^S}{\partial y^2} = \psi^S,$$

$$\psi^S(y = 0) = -\psi^I(y = 0), \quad \psi^S(y \gg \sqrt{R_o}) \rightarrow 0.$$

The solution is

$$\psi^S = y_o e^{-y/\sqrt{R_o}}$$

Transformation (A3) gives:

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{R_o}} \frac{\partial}{\partial \xi}, \quad (\text{A4a})$$

$$\frac{\partial}{\partial y} = -\frac{1}{\sqrt{R_o}} \tan \vartheta \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \zeta}. \quad (\text{A4b})$$

Since $\sqrt{R_o} \ll 1$, we may ignore the $\partial/\partial\zeta$ term of $\partial/\partial y$ in the boundary layer, so we can approximate the Laplacian operator by:

$$\nabla^2 = \frac{1 + \tan^2 \vartheta}{R_o} \frac{\partial^2}{\partial \xi^2}. \quad (\text{A5})$$

Now, Eq. (A1) in the western and eastern boundary layer is:

$$\frac{\partial^2 \psi^B}{\partial \xi^2} = p^2 \psi^B, \quad (\text{A6a})$$

$$\psi^B(\xi = 0) + \psi^I(\xi = 0) = \psi^B\left(\xi = \frac{r'}{\sqrt{R_o}}\right) + \psi^I\left(\xi = \frac{r'}{\sqrt{R_o}}\right) = 0, \quad (\text{A6b})$$

where $p = (1/\sqrt{1 + \tan^2 \vartheta})$. The solution is hence

$$\psi^B = (y - y_o)[-e^{-p\xi} - e^{-p[(r'/\sqrt{R_o}) - \xi]}]. \quad (\text{A7})$$

So the final solution is:

$$\psi = y_o e^{-(y/\sqrt{R_o})} + (y - y_o)[1 - e^{-p\xi} - e^{-p[(r'/\sqrt{R_o}) - \xi]}]. \quad (\text{A8})$$

Note if we set $\vartheta = 0$, we cannot recover the Fofonoff solution in rectangular domain since we have not specified the upper boundary condition. Figure 26b shows the streamfunction contour for the Fofonoff solution with $R_o = 4. \times 10^{-4}$, $y_o = 0.5$, $a = 3.25$.

2. Stommel solution

Here we are concerned with the dimensionless equation

$$\epsilon_s \nabla^2 \psi + \frac{\partial \psi}{\partial x} = -\sin \pi y, \quad (\text{A9})$$

where the nondimensional parameter $\epsilon_s = \nu/\beta$ measures the magnitude of bottom drag relative to the β -effect, and is presumptively small.

Away from the boundary, we can write the solution of (A9) immediately:

$$\psi' = -x \sin \pi y + d(y) \sin \pi y. \quad (\text{A10})$$

Now we do the coordinate transformation:

$$\xi = \frac{x}{\epsilon_s} - \frac{\zeta}{\epsilon_s} \tan \vartheta, \quad (\text{A11a})$$

$$\zeta = y, \quad (\text{A11b})$$

$\epsilon_s \xi$ gives the distance from the western boundary. Now the western and eastern boundary are given by $\xi = 0$ and $\xi = r'/\epsilon_s$, where r' is the width given by $r' = a(1 - \zeta)$.

Under the transformation (A11), we have

$$\psi' = (-\epsilon_s \xi - \zeta \tan \vartheta + d) \sin \pi \zeta. \quad (\text{A12})$$

Requiring $\psi' = 0$ at eastern boundary yields:

$$d = r' + \zeta \tan \vartheta \quad (\text{A13})$$

Transformation (A11) gives:

$$\frac{\partial}{\partial x} = \frac{1}{\epsilon_s} \frac{\partial}{\partial \xi}, \quad (\text{A14a})$$

$$\frac{\partial}{\partial y} = \frac{1}{-\epsilon_s} \tan \vartheta \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \zeta}. \quad (\text{A14b})$$

Again we may ignore the $\partial/\partial\zeta$ term of $\partial/\partial y$ in the boundary layer since $\epsilon_s \ll 1$, so we can approximate the Laplacian operator by:

$$\nabla^2 = \frac{1 + \tan^2 \vartheta}{\epsilon_s^2} \frac{\partial^2}{\partial \xi^2}. \quad (\text{A15})$$

So now the equation (A9) in the boundary layers is:

$$\left(\frac{\partial^2}{\partial \xi^2} + p \frac{\partial}{\partial \xi} \right) \psi^B = 0, \quad (\text{A16})$$

where $p = [1/(1 + \tan^2 \vartheta)]$. The solution is hence

$$\psi^B = [ae^{-p\xi} + be^{-p[(r'/\epsilon_s) - \xi]}] \sin \pi\zeta. \quad (\text{A17})$$

Requiring the boundary conditions that ψ is zero at $\xi = 0$ and $\xi = (r'/\epsilon_s)$ (the southern boundary condition automatically satisfied), we have:

$$a = -r', b = 0.$$

So finally the solution is:

$$\psi = [r'(1 - e^{-p\xi}) - \epsilon_s\xi] \sin \pi\zeta. \quad (\text{A18})$$

Note our derivation is actually independent of the detailed form of wind stress as long as its curl vanishes as it approaches the northern and southern boundaries. Also, if we set $\vartheta = 0$, we recover the Stommel solution in rectangular domain. Figure 26c plots the streamfunction for Stommel solution with $\epsilon_s = 0.01$, and $a = 3.25$.

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