PROBLEMS AND PHENOMENOLOGY IN TWO-DIMENSIONAL TURBULENCE

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Abstract. Various aspects of two-dimensional turbulence are summarized, and certain numerical experiments are described. The successes and failures of 'classical' phenomenology, and the implications for atmospheric dynamics, are discussed.

1. Preamble. This is a short and incomplete but fairly self-contained description of a selection of developments in two-dimensional turbulence, concentrating on those which have occurred over the last decade or so and their relationship to classical or Kolmogorov based phenomenology. It is not a comprehensive review, and the reader will perforce not obtain a completely balanced viewpoint. However, it is hoped that some appreciation may be gained for the current status of the field and some of the outstanding problems. It may be read by the neophyte without referral to the original literature, although not all statements are proven. For background material see Kraichnan and Montgomery (1980) and Lesieur (1987). The goal is to give the simplest discussion of the salient physics of current problems, not for pedagogic reasons but because this is the most effective way to assess the field. Notable omissions are discussions of closure, geostrophic turbulence, vortex dynamics, and multi-fractals.

We first very briefly recall some elementary results in 2-D fluid dynamics and inviscid statistical mechanics. There is nothing new here, and the reader familiar with the material may skip directly to §4, where we review the classical cascade arguments arising from Kolmogorov-Kraichnan-Batchelor-Leith (KKBL) phenomenology for forced viscous problems (Kolmogorov, 1942: Kraichnan, 1967; Leith, 1968; Batchelor, 1969). Then we summarize some of the more recent numerical evidence which seems to throw some doubt on the detailed applicability of the classical phenomenology. We also discuss some numerical work which highlights some successes of classical ideas, and which point to the difficulty one may have in constructing broad new frameworks. Various questions and problems are littered throughout.

2. Basic results of inviscid turbulence

2.1 Existence. The motion of an incompressible, two dimensional fluid is governed by the vorticity equation:

$$\frac{D\zeta}{Dt} = 0 \tag{2.1}$$

or

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0 \tag{2.2}$$

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where $J(\psi,\zeta) = \partial\psi/\partial x\partial\zeta/\partial y - \partial\psi/\partial y\partial\zeta/\partial x$ and ζ is the vorticity and ψ the streamfunction. These are connected by

$$\zeta = \nabla^2\psi = \hat{\mathbf{k}}\cdot\nabla\times\mathbf{u} \tag{2.3}$$

where $\mathbf{u} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}$ is the horizontal velocity and $\{\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}\}$ are unit vectors in the x-, y-, and z-directions respectively.

It has been known for over fifty years (see e.g. Rose and Sulem, 1978) that this equation is well-posed in the following sense. If the initial vorticity field is Hölder continuous, and the initial velocity is C^n, then the velocity field will remain so for all finite time. This result is a global regularity result, and is often referred to as proving the existence of unique solutions to (2.1). A corollary of this is that the solutions at time $t > 0$ depend *continuously* on the data at $t = 0$. The addition of a viscous term $\nu\nabla^2\zeta$ to the right-hand-side of (2.2) is not necessary to prove existence, but nor does it invalidate the result. This result is sufficient to enable us to refer to 2-D turbulence as a dynamical system, whereas no such statement is strictly possible for 3-D turbulence, since the existence of solutions without singularities for neither the Euler equations nor the Navier-Stokes equations has been proven. This has some ramifications for predictability, as discussed in Leith and Kraichnan (1972) and Vallis (1985). Essentially, it means that two-dimensional turbulence is predictable in an 'epsilon-delta' sense, in that an error in prediction at any finite future time may be made as small as we like by making the initial error small enough. This does not mean that 2D turbulence is not chaotic: for a wide parameter range it has positive Lyapunov coefficients and is unpredictable in that sense.

2.2 Inviscid Invariants. Vorticity itself is a Lagrangian invariant, because the form of (2.1) implies that material fluid elements carry their values of vorticity with them. The consequence of this is that any integral function of vorticity is conserved. This actually follows with no equations: If a material element carries its vorticity, then it carries with it any function of the vorticity, and the evolution is merely a re-arrangement of any function of the vorticity. Thus the measure of any value of any function of vorticity is unchanged, and thus by definition the Lebesgue integral of the function over the domain is unaltered.

To see this more conventionally, multiply (2.1) by $G'(\zeta)$, where G is any differentiable function. Then it immediately follows that

$$\frac{\partial G}{\partial t} + J(\psi, G) = 0. \tag{2.4}$$

Integrating over the domain we have, provided there are no boundary contributions, that

$$\int J(\psi, G)\,\mathrm{d}\,\mathbf{x} = 0 \tag{2.5}$$

and therefore

$$\frac{d}{dt}\int G\,\mathrm{d}\,\mathbf{x} = 0. \tag{2.6}$$

(This result requires that G be differentiable.) Note that no relationship between streamfunction and vorticity is used in these proofs. This is related to the fact that when (2.2) is written in Hamiltonian formalism, these invariants are independent of the form of the Hamiltonian, and depend only on the form of the Lie-Poisson bracket. Such invariants are called Casimirs.

A special case of (2.6) is the conservation of enstrophy,

$$\frac{d}{dt}\int \zeta^2 \mathrm{d}\,\mathbf{x} = 0. \tag{2.7}$$

A form of energy is also conserved by (2.2). Explicitly,

$$E = \frac{1}{2}\int (\nabla\psi)^2 \mathrm{d}\,\mathbf{x},\ \ \frac{dE}{dt} = 0. \tag{2.8}$$

3. Statistical mechanics. This is a property of the spectrally expanded version of (2.2). For simplicity, consider a doubly-periodic plane on which (2.2) applies. Then we may expand

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \psi_k \exp(i\mathbf{k}\cdot x) \tag{3.1}$$

and similarly for $\zeta(\mathbf{x})$, where we note the spectral coefficients of ζ and ψ are related by $\zeta_k = -k^2\psi_k$. (Spectral and physical space variables are denoted with the same symbol, being differentiated by their arguments.)

Substituting (2.4) into (2.2) yields the infinite system of ODE's

$$\dot{\zeta}_k = \sum_{pq} A_{kpq}\zeta_p\zeta_q \tag{3.2}$$

where A_{kpq} are geometric coefficients. The precise form of the A_{kpq} is less important, here, than the fact that they vanishes unless

$$\mathbf{k}+\mathbf{p}+\mathbf{q} = 0 \tag{3.3}$$

This leads immediately to the detailed Liouville property,

$$\frac{d\dot{\zeta}_k}{d\zeta_k} = 0 \tag{3.4}$$

This means that the volumes in phase space are conserved, or that the flow in the phase space is incompressible, a necessary precursor to doing almost any sort of statistical mechanics on the system (Tolman, 1938).

Because of the presence of Liouville's theorem and the invariants of motion, it is natural to try to predict ensemble averages on the basis of classical statistical mechanics. The immediate question arises, out of the infinity of Casimir invariants, which ones should be used? The answer is actually determined by the fact that we can only do the statistical mechanics in spectral space for a truncated version of

(2.4): if all wavenumbers are included, the generalized equipartition state we may expect to achieve would imply zero amplitude in each mode. Hence we must do the problem first for a finite number of modes, and then let the truncation wavenumber, k_m, tend to infinity. Only the quadratic invariants survive the spectral truncation—the energy and enstrophy. This is because the energy and enstrophy are conserved triad-wise, meaning that any interacting triad (2.6) will conserve them. However, the non-quadratic invariants plainly do not survive the truncation, because they are not preserved in a single triad. Thus, whereas

$$\int \zeta J(\psi,\zeta)\mathrm{d}\,\mathbf{x} = \sum_{kpq}{}' A_{kpq}\zeta_k\zeta_p\zeta_q = 0 \tag{3.5}$$

where the primed sum indicates that the sum is taken only over modes included in the truncation, no such finite truncation is able to reproduce the result

$$\int G'(\zeta)J(\psi,\zeta)\mathrm{d}\,\mathbf{x} = 0 \tag{3.6}$$

unless $G(\zeta)$ is a linear function of ζ.

Under the conditions of a spectral truncation of (2.4), with energy and enstrophy invariants alone, we may assume the Gibbs distribution

$$P \propto e^{-(E+\alpha Z)} \tag{3.7}$$

Standard methods then lead to the predicted energy distribution

$$U_k = \frac{1}{(a+bk^2)} \tag{3.8}$$

where $U_k = k^2|\psi|^2$ is the energy of mode $\mathbf{k}$. The parameters a and b are Lagrange multipliers, whose values are given by the values of energy and enstrophy. This is sketched in fig. 1, for two values of k_m. As the truncation wavenumber tends to infinity, then it is not difficult to show that $a \to k_m^4/\exp(k_m^2)$ and $b \to k_m^2$, where the values of energy and enstrophy are both taken to be unity. Energy is trapped more and more at the lowest wavenumber, and enstrophy is pushed to the highest wavenumbers. This result is often used to infer the direction of the cascades in forced dissipative turbulence—enstrophy to high wavenumber and energy to small. The physical content is similar to the statement that a given spectrally localized energy distribution will, if the distribution broadens, transfer energy to small wavenumbers and enstrophy to large.

4. Cascade phenomenology. We first note that in the limit of infinite Reynolds number, energy dissipation is zero. The viscous equation of motion is

$$\frac{D\zeta}{Dt} = \nu\nabla^2\zeta \tag{4.1}$$

That is, the value of vorticity is conserved on parcels, except for the action of viscosity which can only act to reduce maximum values. (Actually, we have not

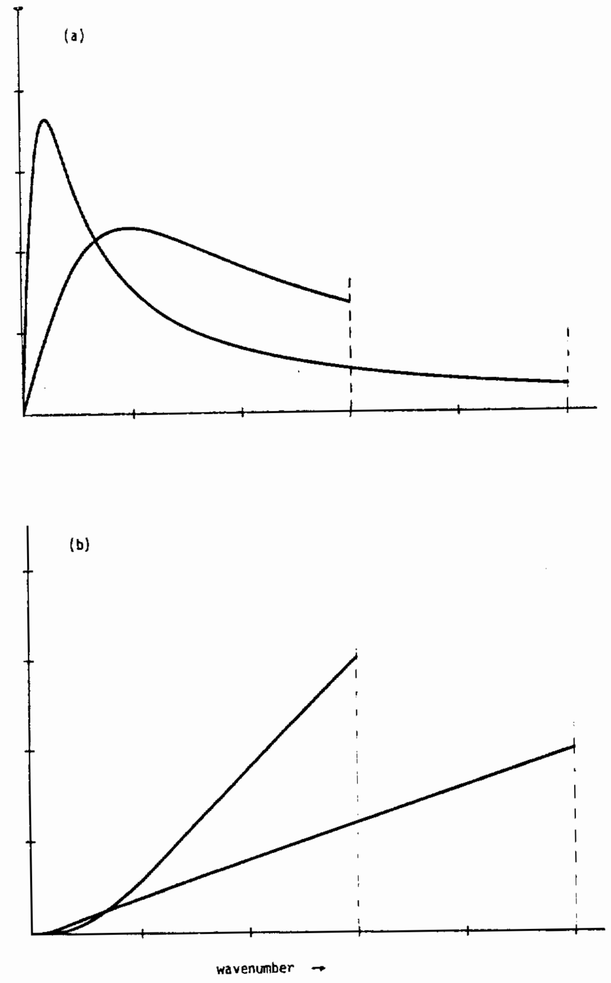


Fig.1 (a) Energy and (b) enstrophy spectra in thermal equilibrium for two diffferent truncation wavenumbers. In the limit of infinite resolution, energy is trapped entirely at the lowest wavenumber, whereas no enstrophy remains at finite wavenumber.

rigorously proven this, but it is almost obvious from (4.1).) From (4.1), we derive the the energy equation

$$\frac{dE}{dt} = \frac{d}{dt}\frac{1}{2}\int (\nabla\psi)^2 \mathrm{d}\,\mathbf{x} = -\nu \int \zeta^2 \mathrm{d}\,\mathbf{x} \tag{4.2}$$

As $R_e \to \infty$, or $\nu \to 0$, energy dissipation can only remain finite if $\int \zeta^2 \mathrm{d}\,\mathbf{x}$ becomes infinite, which in a finite domain requires ζ to be infinite somewhere. This is forbidden by (4.1) (and more rigorously using regularity results) and hence energy dissipation goes to zero. Enstrophy dissipation, on the other hand, is given by

$$\frac{DZ}{Dt} = \frac{d}{dt}\frac{1}{2}\int (\nabla^2\psi)^2 \mathrm{d}\,\mathbf{x} = -\nu \int (\nabla\zeta)^2 \mathrm{d}\,\mathbf{x} \tag{4.3}$$

This may stay finite as $R_e \to \infty$, and indeed under KKBL phenomenology is hypothesized to stay constant, as discussed more below.

In forced dissipative turbulence, we heuristically imagine stirring at some large or intermediate scale, and that dissipation occurs at small scales. In three-dimensional turbulence the sense of energy transfer is unambigously to small scales. However, in two-dimensions energy dissipation vanishes at high Reynolds number. Further, the arguments based on the inviscid statistical mechanical equilibrium suggest that as the truncation wavenumber increases, enstrophy alone cascades to small scales. These considerations, along with the original ones of Kolmogorov (1941), lead one to propose the following phenomenology in homogeneous two-dimensional turbulence (cast in the same lines as Kolmogorovs original assumptions regarding the nature of three-dimensional turbulence, except we use the inertial range flux, η, in place of dissipation rate, ϑ):

(0) Given a spectrally local energy and enstrophy source (i.e. it has spectrally compact support) we assume that there is a finite flux of enstrophy to small scales and energy to large. In a steady state the enstrophy flux is equal to enstrophy dissipation. (*Corollary:* Without introducing artificial viscosities, the energy flux will keep on cascading to larger scales. In a finite domian it will accumulate at the largest scale.)

(1) In the scales smaller than the forcing scale the energy spectrum $E(k)$ (or more generally the n-variate probability distributions for the velocity differences $\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$) is assumed to be a universal function of the mean enstrophy flux η, the kinematic viscosity ν, and the wavenumber k (or the difference vector $|\mathbf{r}|$).

(2) If the wavenumbers k are large in comparison to the dissipation scale, then the energy spectrum is independent of ν. This is equivalent to assuming that the enstrophy flux through a wavenumber k depends only on local quantities, namely the wavenumber and energy $\mathcal{E}(k)$ or velocity $v(k)$, and is not a direct function of wavenumber. This is essentially the locality hypothesis.

(3) In the scales significantly larger than the forcing scale, the energy spectrum is a universal function of the energy flux, ϵ and wavenumber.

Assumption (0) is unique to two-dimensional turbulence. Assumption (1) is analogous to Kolmogorov's first hypothesis, and (2) Kolmogorov's second hypothesis. Assumption (3) is equivalent to (2), but applied to the energy cascading range. Note that energy dissipation is not a meaningful quantity, and the energy flux through the energy range must clearly be used in its place. Indeed it is likely that the flux is a more central quantity to the dynamics than dissipation for the enstrophy range also—since at high Reynolds numbers the enstrophy dissipation is driven by the flux through the equilibrium range. Certainly we may also imagine, in a thought experiment, a fluid of extremely high ('almost infinite') Reynolds number in which the high wavenumber end of the enstrophy range is still extending itself, and has not yet reached the dissipation range. There is nevertheless a flux of enstrophy to small scales, and an enstrophy equilibrium range, but virtually no enstrophy dissipation. It seems likely that the dynamics in the inertial range is essentially the same as that for the finite Reynolds number case. If this is so, it implies that the enstrophy flux is the dynamically important quantity in the enstrophy range. Such a thought experiment is however not possible for the 3D case, in which the dissipation scale is reached in finite time no matter how small the viscosity.

In the energy range, it makes little sense to invoke a hypothesis akin to (1) by itself, since energy dissipation is not dynamically important. Thus hypothesis (3) is immediately proposed. (One might have hypothesized a dependence on forcing type, as indeed one might have for the enstrophy range). Indeed, the assumption that the energy spectrum in the enstrophy (energy) range is independent of the energy (enstrophy) flux already implicitly assumes some degree of locality. In general the *utility* of the KKBL theory will depend very much on the extent to which the locality hypothesis is satisfied and the energy spectra are independent of the details of the forcing and dissipation. The conditions under which it holds are the subject of continuing investigation and controversy in both two- and three-dimensional turbulence.

Dimensional analysis may then be used to ascertain the functional forms of the spectra in the two ranges. Such an analysis is closely related to the use of the scale invariance of the equations, as follows. In the absence of viscosity, the equation of motion (2.2) is invariant under the following scale transformation:

$$x \Longrightarrow x\lambda \; v \Longrightarrow v\lambda^r \; t \Longrightarrow t\lambda^{1-r} \tag{4.4}$$

where r is an arbitrary scaling exponent. Kolmogorov's first and second hypotheses are essentially equivalent to assuming that the *dynamics* obeys the scale invariance (4.4), on a time-average, in the intermediate scales between the forcing scales and dissipation scales.

Dimensional analysis then tells us that the energy flux ϵ scales as

$$\epsilon \sim \frac{v^3}{l} \sim \lambda^{3r-1} \tag{4.5}$$

from which the assumed constancy of ϵ gives $r = 1/3$. The velocity scales as $v \sim \epsilon^{1/3}k^{-1/3}$ Then we obtain a prediction for the energy spectrum:

$$E(k) \sim v^2k^{-1} \sim \epsilon^{2/3}k^{-2/3}k^{-1} = K\epsilon^{2/3}k^{-5/3} \tag{4.6}$$

where K is a dimensionless constant. In the Kolmogorov theory the slope of the energy spectrum k^{-n} is related to the scaling exponent r by $n = -(2r+1)$.

The enstrophy flux η scales as

$$\eta \sim \frac{v^3}{l^3} \sim \lambda^{3r-3} \tag{4.7}$$

which from the assumed constancy of η in the enstrophy range gives $r = 1$, and an associated energy spectra of

$$E(k) = K'\eta^{2/3}k^{-3}, \tag{4.8}$$

where $\mathcal{K}'$ is a constant. In (4.6) and (4.8), ϵ and η are averages. That is, $\eta^{2/3} = \overline{\eta}^{2/3}$ and not $\overline{\eta^{2/3}}$, where an overbar denotes an average.By convention the overbar is omitted where it will cause no confusion.

At small scales, dissipation must become important. Dimensionally, we expect this to occur at the inner scale l_ν given by

$$l_\nu \sim \left(\frac{\nu^3}{\eta}\right)^{1/6} \tag{4.9}$$

The structure functions S_m of order m, which are the average of the m'th power of the velocity difference over distances $l \sim 1/k$, scale as $(\delta v_l)^m \sim l^{rm} \sim k^{-rm}$. In particular the second-order structure function, which is the Fourier transform of the energy spectra, scales as $S_2 \sim k^{-2r}$, for $r \leq 2$.

Before discussing questionable aspects of the phenomenology, we mention one aspect sometimes thought to be a problem, but which is not. Enstrophy dissipation scales as

$$\begin{aligned} \dot{Z} &= \frac{d}{dt}\int \zeta^2 dx = \nu \int \zeta \nabla^2 \zeta \\ &\sim \nu k_\nu^4 v^2 \sim \eta \end{aligned} \tag{4.10}$$

where $k_\nu \sim 1/l_\nu$. This scaling must hold even in the infinite Reynolds number, or zero viscosity, limit. However, it is known rigorously that enstrophy dissipation is zero for zero viscosity, and that there are no singularities in the vorticity gradient field. Is this not inconsistent? No. Viscosity becomes important at wavenumbers smaller than k_ν. If the Reynolds number is increased slowly, allowing the fluid time to equilibrate at each new value, the enstrophy dissipation will indeed stay constant although it takes an infinite time to achieve an infinite Reynolds number. If we 'turn-off' viscosity, it also takes an infinite time for the enstrophy inertial range to extend itself into the dissipation scale, and only then may dissipation resume. For all finite time, the dissipation is zero. The point is that the phenomenology takes an infinite time to establish itself (unlike in three dimensions); for finite time at infinite Reynolds number there is no enstrophy dissipation, and no contradiction with the regularity results.

Two aspects of the phenomenology are of immediate concern, namely nonlocality and intermittency.

4.1 Nonlocality. If the locality assumption is not made, then in the enstrophy range the energy spectrum may take the general form

$$E(k) = \mathcal{K}'\eta^{2/3}k^{-3}f(k/k_o)g(k/k_\nu) \tag{4.11}$$

where k_o and k_ν are the forcing and dissipation scales, respectively. Such an equation is no use if f and g are arbitrary. If the locality hypothesis is satisfied, f and g are eliminated and the spectral slope is k^{-3}.

In addition to the obvious requirement of a local step-wise cascade of energy or enstrophy discussed more below, an additional, more subtle, requirement for locality is chaos. For even if the enstrophy cascade to small scales proceeds by relatively local triad interactions, if there is no chaos then at each step of the cascade a memory of the large scale forcing will be retained. Indeed this has been hypothesized to be the cause of the -2 spectra in Burgers 'turbulence'. Here the governing equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

From smooth initial data, it can readily be shown that shocks form in the u-field, and energy cascades to small scales where it is dissipated. Because of the formation of shocks, the energy spectrum is proportional to k^{-2}. But if there is an energy cascade, should not Kolmogorov scaling hold and the energy spectrum be proportional to $k^{-5/3}$? The resolution of this paradox is that Burgers equation is not chaotic. The Cole-Hopf transformation (see e.g. Whitham, 1974) renders (4.10) into a linear heat conduction problem, for which the solution may be found analytically. Thus the energy cascade is not chaotic; the energy flux may 'remember' the largest scales, $f(k/k_o)$ is not unity, and a -5/3 spectrum is not required.

Two-dimensional turbulence, whilst it may be well-posed, is certainly chaotic. It therefore seems unlikely that a local enstrophy or energy cascade could proceed without producing KKBL spectra. Is the cascade process in fact local? Given such KKBL spectrum, we may heuristically estimate whether, *a posteriori,* the enstrophy range spectrum is local and thence self-consistent. To estimate this we calculate the contribution to strain at a waveumber p from all other smaller wavenumbers. The strain, $S(k)$is given by:

$$\begin{aligned} S(k) &= \int_{k_o}^{k} E(p)p^2 dp \\ &\sim \int_{k_o}^{k} \eta^{2/3} d\ln p \end{aligned} \tag{4.12}$$

This indicates that for a -3 spectrum, each octave makes the same contribution to the strain integral, implying the spectrum is hardly local after all. Although this is not a rigorous result, or even a closure based result, it is likely that it is the root cause of all the 'problems' associated with the non-compliance of numerical simulations with KKBL phenomenology, as described below. A related aspect of this problem is as follows. Self-consistency demands that the characteristic timescales of the small scales be small compared with the large scales, in order that they may remain in equilibrium with the large scales. Dimensionally, an eddy-turnover timescale for a scale k is simply

$$\tau(k) \sim \eta^{-1/3}$$

which plainly does not decrease with decreasing scale. Even a slightly 'improved' estimate of this, namely

$$\tau(k) = \left[\int_{k_o}^{k} p^2 E(p)\, dp\right]^{-1/2} \tag{4.13}$$

decreases only logarithmically with wavenumber.

Although these considerations are heuristic, explicit numerical computations by Ohkitani (1990) and Maltrud and Vallis (1991) (henceforth MV) do indicate that enstrophy transfer in the inertial range is quite non-local, in the sense that very elongated triads dominate the palinstrophy production. The upshot of these considerations is that the -3 slope is not so much the likely outcome of an experiment, but the shallowest slope which can be achieved. For if the spectral energy slope is shallower than -3, most of the contribution to the integral in (4.12) comes from the neighbourhood of p, implying locality. In that case, the dimensional arguments following (4.4) must apply and a slope shallower than -3 is inconsistent. However, given a spectral slope steeper than -3, the contributions to the integral in (4.11) come mainly from the low-wavenumber end. The transfer is therefore highly non-local, and the freedom to choose arbitrary dimensionless functions in (4.11) means that such spectra are by no means inconsistent.

This apparent inconsistency in the strict -3 spectra led Kraichnan (1971) to propose a log-corrected range. The reasoning goes as follows. Dimensionally, the enstrophy flux is the ratio of the enstrophy to a time. That is,

$$\eta = C\frac{E(k)k^3}{\tau(k)} \tag{4.14}$$

where C is a dimensionless constant. To estimate $\tau(k)$ we use (4.13), which yields

$$E(k) = K'\eta^{2/3}k^{-3}(\ln(k/k_o))^{-1/3} \tag{4.15}$$

The log-correction, even if present, is likely to be observationally indistinguishable from a true -3 range. It should be emphasized that (4.15) *does not* unambiguously overcome the problems associated with non-locality in the enstrophy range. For (4.13) recognizes non-locality, and proposes a phenomenological solution. In effect, the scheme predicts $f(k/k_o) = (\ln(k/k_o))^{1/3}$ and $g(k/k_\nu) = 1$. Enstrophy transfer in the log-corrected range is still non-local. However, with the hindsight of numerical simulations discussed below, there seems no compelling reason why such a form should be chosen by the dynamics. For example, if the energy spectrum is aware of the low wavenumbers, then why should the energy spectrum not also depend on the form of the forcing? To paraphrase Kraichnan himself in a slightly different context (Kraichnan, 1974), once we abandon the Kolmogorov 1941 theory a Pandora's box of possibilities is opened.

In contrast to the -3 case, the $-5/3$ energy spectra is phemonenologically more local and self-consistent, although closure estimates indicate the energy transfer is much less local than in the -5/3 equilibrium range in three dynamics (Kraichnan, 1971). Numerical experiments show it to be a quite robust feature of two-dimensional turbulence.

4.2 Intermittency. Intermittency in general refers to the nonconstancy of energy of enstrophy transfer or dissipation. The general area of intermittency in turbulence is too large to summarize concisely, and when restricted to two-dimensional turbulence it is too new and involved for a clear picture of its dynamical effects and importance to have emerged. A full discussion leads into the burgeoning theory of fractals and multi-fractals (Paladin and Vulpiani, 1987; Warn, 1991). This section is therefore likely to rapidly become outdated, and probably already is. We restrict attention to the forward enstrophy cascade and make a few general remarks.

There is no difficulty with KKBL phenomenology if the cascades rates are constant in space and time. This requirement seems unlikely to be achieved in practice. In a particular realization of a (homogeneous) ensemble the intensity of the small scale turbulence is likely to depend on the intensity of the local forcing (Landau and Lifshitz, 1987; Warn 1991), and global universality will fail. The essential problem is that the energy spectra is a nonlinear function of the enstrophy flux η, or dissipation ϑ and

$$\bar{\eta}^{2/3} \neq \overline{\eta^{2/3}} \tag{4.16}$$

(Note that $\eta = \vartheta$, but that higher powers of these two quantities need not be equal.) The left hand side of (4.16) is an average whereas the right-hand-side is the 2/3 power of an average.

This particular objection is overcome by a modified or refined theory (Kolmogorov, 1962). The essential aspect of this theory (as applied to the enstrophy range of 2-D turbulence) is that the enstrophy dissipation, ϑ, is replaced by a 'coarse-grained', or spatially averaged, dissipation obtained by averaging ϑ over a small area of size $a \sim 1/k$. If this is ϑ_a then the prediction of the energy spectrum becomes

$$E(k) = K\overline{\vartheta_a^{2/3}}k^{-3} \tag{4.17}$$

However, this seems unsatisfactory (also see Kraichnan, 1974). First, attention is drawn to the dissipation field, rather than the flux through the inertial range which physically seems more important. Whereas their means must be equal, their higher order statistics need not. Second, it is arbitrary, or at best *ad hoc.* This applies in particular to Kolmogorov's rather specific assumptions about the log-normality of the energy dissipation (in three dimensional turbulence).

In any case, the modified Kolmogorov theory does not fundamentally change the picture of a succession of random cascade steps. In three-dimensional turbulence the $-5/3$ law seems experimentally well satisfied, but higher order structure functions do not obey Kolmogorov scaling (e.g. Anselmet, 1984). In two dimensions the energy slope itself does not appear to follow the Kolmogorov form (see §5), and it is currently not known how the higher order structure functions behave. There has naturally been much interest in determining how intermittency may modify the energy slope and higher order structure functions. Various models have attempted to include the effects of a fluctuating energy dissipation, or intermittency, including the beta-model of Frish *et al.* (1978) based on the Novikov-Stewart model (Novikov and Stewart, 1964). The idea of the beta model is that dissipation is not space-filling, and at each 'step' in the cascade process the energy transfer is confined to a smaller region of space. This affects the phenomenology for two-dimensions, as follows (Basdevant *et al.*, 1981).

If the turbulence is confined to an 'active' sub-domain, the energy spectrum in the active sub-domain, call it $\hat{E}(k)$, determines the important dynamical quantities like eddy-turnover time. Thus,

$$\tau(k) = \left\{ \int p^2 \hat{E}(p) dp \right\}^{-1/2} \tag{4.18}$$

If we suppose that the active spectrum at each scale occupies a sub-domain proportional to $(k/k_o)^{-\gamma}$, then we have

$$E(k) = (k/k_o)^{-\gamma} \hat{E}(k) \tag{4.19}$$

Using (4.18) and (4.19) in the formula

$$\eta = \frac{k^3 E(k)}{\tau(k)} \tag{4.20}$$

we obtain an energy spectrum given by

$$E(k) \sim k^{-3-\gamma/3}, \tag{4.21}$$

which is steeper than -3, and an active spectrum given by:

$$\hat{E}(k) \sim k^{-3+2\gamma/3} \tag{4.22}$$

which is shallower than -3. In this theory, $2-\gamma$ may be interpretated as the fractal dimension of the dissipative structures (and therefore $0 \leq \gamma \leq 2$). The steepest spectrum possible is therefore -11/3, although if intermittency in time is allowed further steepening may occur.

Benzi *et al.* (1986) argue that, in fact, beta models (except for the special case of Kolmogorov's law) cannot be applied to two-dimensional turbulence, because for nonzero γ such models lead to singularities in the velocity field, which are forbidden by regularity. In other words, enstrophy dissipation must be space filling and cannot be confined to a fractal measure. The putative singularities arise because the active spectra is shallower than -3: From (4.22) we obtain

$$\frac{\delta v(r)}{r} \sim \frac{1}{r^{\gamma/3}} \tag{4.23}$$

which is singular as $r \to 0$. However, the singularity only arises in the limit $r \to 0$. It does not seem logically forbidden that in the equilibrium range the enstrophy flux be confined to a fractal dimension, and the beta-like models may be applicable there. In the dissipation range the energy spectrum may be quite steep, and structures space filling with an 'active' spectrum steeper than -3. It is in any case likely that the application of fractal and multi-fractal ideas in two dimensional turbulence will continue (e.g. Mizutani and Nakano, 1989). Certainly, their use must also be reconciled with the appearance of isolated vortices, as Benzi *et al.* (1986) have already attempted. One hopes that such ideas can be used to give specific, testable predictions rather than simply providing a qualitative framework in which to view the subject.

5. Experimental results

5.1 Decaying Turbulence. The first indications that the enstrophy inertial range was not as predicted by classical phenomenology came with the early numerical integrations of two-dimensional turbulence. Although the phenomenology may strictly only be applied to forced-dissipative turbulence, for only then may a statistically steady state be achieved, simulations of turbulence decaying under the action of a weak dissipation display the intrinsic dynamics most cleanly. Simulations such as those of Fornberg (1977) and, most clearly, McWilliams (1984) show clearly the emergence of isolated, long-lived vortices from structureless initial conditions (Fig.2, see color insert). They are 'long-lived' because their lifetime is much longer than a typical eddy turnover timescale at a scale defined, for example by their diameter. The dynamics of their emergence is described elsewhere (e.g. McWilliams, 1990), but, having emerged, their destruction is only effected by collision with another vortex. Indeed, the motion of the vortices is determined largely by the other vortices, just as in the motion of point vortices. This was clearly demonstrated by Benzi *et al.* (1987a). At a certain point in an integration of decaying turbulence after the emergence of isolated vortices, a parallel point vortex simulation was begun, with the strengths and initial positions of of the point vortices determined from the full simulation. For a number of eddy turnover times, the motion of the point vortices resembled quite closely that of the vortices in the original simulation.

The investigation of vortex dynamics continues for its own ends. Recently the spontaneous emergence of dipoles and tripoles has been reported (Legras *et al.*, 1988), although whether they are intrinsically important or whether they are improbable chance occurrences with little effect on the fundamental dynamics is not clear. For our purposes, the issue is only how vortex emergence affects the classical phenomenology.

5.2 Forced-Dissipative Dynamics. Whereas decaying turbulence may be useful for elucidating the dynamics in its purest form, only in forced-dissipative turbulence can a statistically steady state be achieved and hence clean comparisons with the predictions of classical phenomenology made. Currently (c. 1991), simulations can routinely be performed at resolutions of about 512^2, although much higher resolution can be achieved in special circumstances or for isolated experiments. Whereas this resolution would not be enough to unambiguously confirm or falsify various subtle predictions about the precise slope of the energy spectra or the value of Kolmogorov's constant, or even if it exists, it is certainly adequate to indicate clear failures and successes.

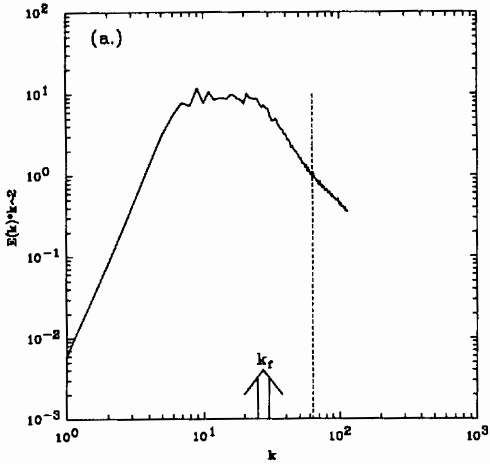
A typical experiment consists of forcing in some fairly localized spectral region, and removing enstrophy at high wavenumbers with some form of viscosity. The form of the viscosity has been found not to be a crucial aspect (McWilliams, 1984, and others), and to achieve as high a Reynolds number as possible 'hyper-viscosities' are used. These take the form $\nu(-1)^{n+1}\nabla^{2n}\zeta$ with typical values in simulations of $n = 2$ or 4; normal molecular viscosity has the value $n = 1$. The forcing is typically a white-noise or markovian, or a negative viscosity (an instability forcing), at much larger scales than the dissipation scale. Since the forcing introduces energy as well as enstrophy, energy must be removed at large scales. This is usually achieved either with a Rayleigh drag confined to small wavenumbers, or with a viscosity proportional to an inverse Laplacian. To achieve the maximum resolution possible in a single experiment, the forcing range is either chosen to be at high wavenumbers, leading to a well-resol ved energy equilibrium range, or at low wavenumbers, leading to as well-resolved as possible enstrophy equilibrium range.

At this resolution, it has become clear that the dynamics of the enstrophy range is indeed influenced by the underlying presence of coherent vortices. One gross but objective measure of the presence of coherence in a field is the vorticity kurtosis, defined by $Kr = < \zeta^4 > / < \zeta^2 >^2$, where $<>$ denotes a spatial average. The kurtosis is a rough measure of the ratio of the distances between the vortices to the vortex size. Even with nearly white-noise forcing (in time) the vorticity kurtosis builds to values considerably larger than the Gaussian value of three. Vortices are typically maintained at or somewhat smaller than the forcing scale. There is some, but not conclusive, evidence that the coherent structures persist only to a scale somewhat larger than the dissipation scale (Legras *et al.,* 1988; MV). In some experiments it has been noted that the spectral energy slope seems to shallow at a wavenumber intermediate between the forcing scale and the dissipation scale. Now, we may calculate the contributions to the kurtosis from a given range of wavenumbers by spectrally truncating the vorticity field, calculating the associated physical space field, and then calculating its kurtosis. We thus define the kurtosis 'as a function of wavenumber', i.e. $Kr(k)$ by including spectral contributions for all wavenumber less than k. This is plotted in fig. 3, where it seen that $Kr(k)$ increases until just that wavenumber at which the energy spectrum begins to shallow. However, at this stage it has not been ascertained definitively how robust these results are, and what role resolution and viscosity play. The addition of differential rotaion to such simulations is also quite effective at reducing the overall kurtosis, although zonal structures may begin to form when differential rotation rates are high (MV).

Nonlocality

The dynamics of the enstrophy range is certainly non-local, by which is meant that the dynamics of the inertial range is aware of the nature of the forcing at much smaller wavenumbers. This is made very apparent in experiments in which the forcing is such as to introduce anisotropy at the largest scale, but in which the dynamics of the inertial range are strictly isotropic. A very effective way of doing this is to include an artificial 'β−effect'[1], which applies only in the limited wavenumber regime in which the isotropic energy and enstrophy producing forcing is being applied. The two classes of forcing, taken together, may be considered as a single anisotropic forcing. In the classical picture, the enstrophy cascade proceeds by a series of small steps, the memory of the forcing details being slowly lost. Thus, one may expect that the small scales would gradually become more isotropic. However, this is not seen at all—the small scales remain anisotropic. It is hard to envision a purely local, chaotic cascade which results in no return to isotropy

[1]The β− effect here, due to differential rotation, is completely unconnected with the beta models referred to earlier. To differentiate, the effects of rotation are denoted with a Greek β.



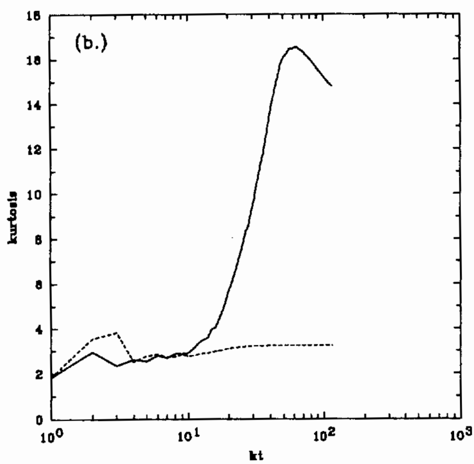


Fig. 3. (a) A typical average enstrophy spectrum ($k^2E(k)$) from a simulation forced in the range 25 $\leq$ k $\leq$ 30 (denoted by k_f). The slope of the lower wavenumber range (approximately 30 $\leq$ k $\leq$ 60) is -2.5, the slope of the higher wavenumber range (approximately 60 $\leq$ k $\leq$ 110) is -1.6. The dashed line indicates the peak (k = 63) in the kurtosis plot in (b). (b) The kurtosis of the vorticity field (solid curve) and stream function (dashed curve) reconstructed using only modes 1 through k plotted as a function of k. (From MV.)

at the small scales. Closure arguments may provide some quantitative insight into this phenomena. Herring (1975) shows that the dynamics may be partitioned into a return-to-isotropy tendency, plus an anisotropy producing aspect. Herring's argument may be paraphrased as follows: The production of anisotropy is largely due to the non-local straining of small scales by the anisotropic large scales, while the isotropizing tendency is largely by local interactions. The Test Field Model gives the following phenomenological estimates for these:

$$\text{Production} \sim \int_0^k \frac{p^2 E_a(p)}{\tau(p)} dp \; E_i(k) \tag{5.1}$$

and

$$\text{Destruction} \sim \tau(k) E_a(k) \tag{5.2}$$

where

$$\tau(k) = \{ \int_0^k p^2 E(p) \, dp \}^{-1/2} \tag{5.3}$$

Here $E_a(k)$ and $E_i(k)$ are the anisotropic and isotropic energy spectra, respectively, and $\tau(k)$ is an eddy turnover time. If the production and destruction terms are in balance, then $E_a(k) \sim 1/\tau(k)$. The important point is that if $E_i(k)$ is k^{-3} or steeper, then $E_a(k)$ does not decrease significantly with wavenumber. On the other hand, for three dimensional turbulence, where $E \sim k^{-5/3}$, E_a will indeed fall off with wavenumber. These results may be thought of as a consequence of the basic result of nonlocality in spectra steeper than k^{-3}.

Intermittency

In decaying non-rotating simulations, the forward enstrophy transfer is almost completely inhibited where isolated vortices form. Indeed, phase scrambling leads temporarily to an enhanced cascade (McWilliams, 1991). Thus a form of intermittency arises simply from the presence of isolated vortices. When vortices collide and merge, there will be a rapid cascade of enstrophy to small scales, and presumably a corresponding burst of enstrophy dissipation—'presumably' because I am not aware of numerical experiments confirming this. This form of intermittency has, superficially, little to do with non space-filling structures and fractal dimensions. It is also associated with a degree of structure less than hospitable to the application of closure theories and classical phenomenology (Herring and McWilliams, 1985).

White noise random forcing acts to inhibit the formation of isolated vortices and reduce this form of intermittency. Indeed Herring and McWilliams (1985) found that in such circumstances moment closure (TFM) performs fairly well. Dispersive wave dynamics (e.g. Rossby waves due to a $\beta-$effect) also act to inhibit the formation of coherent structures and this form of intermittency (Holloway; 1984, MV), and in such circumstances moment closure may also be expected to succeed reasonably well (see also Bartello and Holloway, 1991). But if differential rotation is sufficiently strong so as to engender the formation of zonal jets, moment closure may again be in trouble, although this case has not been extensively investigated.

In the areas between the vortices, the cascade continues in a form more akin to classical phenomenology. Intermittency of the kind discussed in §4.1, which is not necessarily associated with obvious visible structures, may arise. The spectrum in these regions may be expected to be closer to the phenomenological predictions, with small intermittent or other corrections. Indeed, just this was found by Benzi *et al.* (1986), who found a -3 spectrum for the fluid between the vortices whereas the spectrum of the coherent vortices themselves was found to be rather steeper, typically closer to -5. How wave dynamics affects intermittency between the vortices is not well known, and may be moot because of the large effects on the vortices themselves.

A final point: In those situations in which vortices do dominate, the intermediate field may be interpreted as being passively advected by the velocity field set up by the intense vortices, and any intermittency will be similar to that of a passive scalar. As we discuss in the next section, passive scalars appear to exhibit fairly robust -1 spectra, and where enstrophy is behaving passively we obtain a -3 energy spectrum. Thus intermittent corrections may have to be small.

5.3 Passive Scalar Dynamics

A passive scalar tracer, call it ϕ, in two-dimensional dynamics obeys the same equation as the vorticity equation, namely

$$\frac{\partial \phi}{\partial t} + J(\psi, \phi) = F - D \tag{5.4}$$

where F and D represent forcing and dissipation. Since we no longer require $\phi = \nabla^2 \psi$, (5.4) is in a sense more general than the vorticity equation. If $D = -\kappa \nabla^2 \phi$, then the Peclet number UL/κ is analogous to the Reynolds number. Thus, if a simulation is begun with *exactly* the same intial conditions on ϕ and vorticity, and with the same forcing and dissipation, then the two fields will remain identical, with necessarily the same spectral slope, for all time. This will not of course hold generally, although a simple phenomenological argument suggests that a -1 spectrum will be hard to avoid. (Note that a -3 energy slope corresponds to a -1 enstrophy slope)

We suppose that tracer is supplied at some large scale (low wavenumber). In the inviscid case the conserved quantities are of the form $\int G(\phi) \mathrm{d}\, \mathbf{x}$, where $G(\phi)$ is any sensible function. The only quadratic invariant, $\int \phi^2 \mathrm{d}\, \mathbf{x}$ is therefore analogous to enstrophy, and we therefore expect a cascade of tracer concentration to small scales. In the tracer inertial range the transfer of tracer, call it χ, is assumed independent of wavenumber and equal to the rate of its dissipation. Thus,

$$\chi \sim k \frac{\Phi(k)}{\tau} \tag{5.5}$$

where τ is given by (4.13), and $\Phi(k)$ is the squared spectral amplitude of ϕ, so that $\int \phi^2 \mathrm{d}\, \mathbf{x} = \int \Phi(k) dk$. In a log-corrected -3 energy range we obtain $\tau(k) \sim \left(\ln(k/ko)\right)^{-1/3}$, which leads to

$$\Phi(k) \sim k^{-1} \left(\ln(k/ko)\right)^{-1/3} \tag{5.6}$$

If the energy spectrum is steeper than -3, say it is -n, then from (4.13) for $k >> k_o$, $\tau(k) \sim k_o^{(n-3)/2}$. In other words, because the contributions to the strain integral comes mainly from the large scale, $\tau(k)$ is a constant. The passive scalar spectrum is now

$$\Phi(k) \sim k^{-1}, \tag{5.7}$$

(This is identical to the sometimes called Batchelor spectrum, which is the spectrum of a passive scalar beyond the Kolmogorov dissipation range but prior to the tracer dissipation scale, a range which exists if the Peclet number exceeds the Reynolds number). The point is that the passive scalar spectrum is rather insensitive to the energy spectrum, and unlike the case for vorticity where it is easy to envision consistent circumstances leading to a steep spectrum, the -1 spectrum is rather robust.

Babiano *et al.* (1987) attribute the shallower spectrum of a passive scalar to the lack of the presence of intense vortices in the tracer field, because the tracer, being passive, is unable to concentrate itself like vorticity. This point of view has been questioned as being artifactual (Holloway and Bartello, 1991). The point of Holloway and Bartello is that differences in the vorticity and passive scalar dynamics arise even in the absence of isolated vortices in the vorticity field, and furthermore that such differences are in fact quite well predicted by closure (in this case, TFM), which takes no explicit account of isolated vortices. In this interpretation, the shallow tracer spectrum arises, at least in part, from the efficiency of wavenumber local interactions (see also Holloway and Kristmansonn, 1984). Certainly, the cause and effect relationship between the spectral slope and the physical space dynamics of a passive scalar field has not been unambiguously determined (as it has not for the vorticity dynamics). Does the apparent phenomenological necessity of a -1 spectrum somehow forbid the formation of intense 'passive tracer vortices'? Or is their absence primarily because of the dynamical reasons outlined in Babiano *et al.*, enabling classical phenomenology to work? Or are the isolated vortices a complete red herring here?

5.4 Energy Inertial Range. At scales larger than a forcing scale energy is expected to cascade to large scales. Numerical experiments confirm this is the case. Preliminary analyses of these results also indicate that the energy transfer in the -5/3 range is more local than enstrophy transfer in the -3 range.

It appears in fact that the inverse energy cascade is rather robust, with an energy spectrum close to -5/3 and an apparently more-or-less universal value of the Kolmogorov constant of between 6 and 10. Numerical results have been obtained by Frisch and Sulem (1984) and, at much higher resolution, by Maltrud and Vallis (1991), with consistent results (fig.4).

The role of isolated vortices appears, at least superficially, to be less important in the energy range than in the enstrophy range. In a number of numerical experiments at 512^2, MV applied a stochastic forcing confined to wavenumbers close to 180. The energy dissipation was a Rayleigh drag applied only to wavenumbers less than 10. Thus the wavenumber range between 10 and 180 is truly 'inertial'. A well resolved energy inertial range with a spectrum very close to -5/3 was obtained. Visual

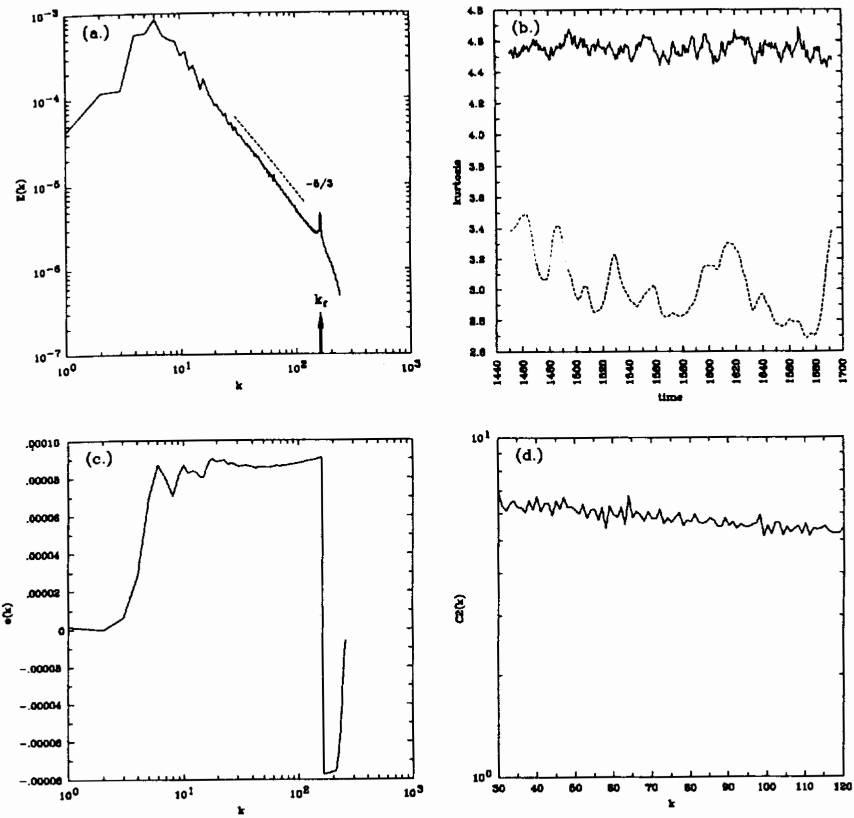


Fig.4. Flow diagnostics from a 512^2 simulation forced in the range $160 \leq k \leq 165$. (a) Time averaged energy spectrum. (b) Time series of vorticity kurtosis (solid curve) and stream function kurtosis (dashed curve) from a statistically steady portion of the run. (c) Time averaged spectral energy flux, $\epsilon(k)$. (d) The Kolmogorov constant $C_2(k) = k^{5/3}E(k)\varepsilon(k)^{-2/3}$. The average value of C_2 is 5.8. (From MV)

inspection of the vorticity field indicated, perhaps surprisingly, no sign of isolated vortices in the energy inertial range, and a corresponding value of kurtosis close to the Gaussian value of 3. It is surprising because it has sometimes been thought that the inverse energy cascade occurs through vortex merger. Instead, we see a process more akin to vortex 'clumping'. Of course in a spectrum as shallow as -5/3 it is rather difficult to imagine isolated vortices of the kind found in the enstrophy range, since rapid variations in vorticity at the edge of the vortices naturally give rise to steep spectra, along the lines of Saffman's argument (Saffman, 1971) that vortex discontinuities give a -2 spectrum of vorticity and a -4 energy spectrum. The presence of *any* kind of large scale structure in the inverse energy cascade remains open. Two issues are outstanding: First, what is the role of the forcing type? MV used a stochastic forcing, with little correlation in time—in some sense a 'strong' forcing because the forcing timescale is shorter than a dynamical timescale. A 'slower' forcing, either a negative viscosity or a redder stochastic forcing might allow the dynamics of the energy inertial range to organize itself through vortex merger. On the other hand, the rather good agreement with phenomenological predictions for those expeiments which have been done implies that the inertial range dynamics may indeed be independent of the details of the forcing. The second issue is more subjective, involving recognition of structures in the inertial range. Perhaps structures exist, but simply cannot be seen in the vorticity field by eye. Only the Shadow knows.

5.5 Dual Forcing Regimes and Atmospheric Dynamics

In atmospheric dynamics the spectral slope of energy is observed to go from a -3 slope for scales larger than about 500km to a -5/3 slope for smaller scales (Nastrom and Gage, 1985). At large scales baroclinic instability produces an effective forcing scale of a few thousand kilometers, at which two-dimensional turbulence is perhaps an acceptable lowest order model of the dynamics. We therefore expect an enstrophy cascade to small scales, and indeed the -3 slope has been thought to be the natural consequence such a cascade, and an observational verification of a basic result of two-dimensional turbulence. However, in the light of newer numerical simulations which tend to produce steeper slopes, this slope actually becomes something of a mystery—why is the atmosphere itself is so close to a -3 slope when simulations of geostrophic turbulence are not? It is especially germane when one considers that the driving of the atmosphere is an instability forcing, more akin to a negative viscosity than a stochastic forcing, and instability forcings tend to give the steepest spectral slopes. The answer plainly does not lie in in the fact that the atmosphere is a continuous fluid and simulations are done on a finite grid, for the atmosphere is only two-dimensional over a finite range. The answer may lie in the phenomenology: a -3 slope tends to arise in the presence of a time independent eddy turnover time. In the atmosphere time independence may come from shear instabilities, or from surface drag, or any number of phenomena. Alternatively the inhomogeneity plus anisotropy (the $\beta-$effect) may act as a destructive influence on nascent isolated vortices (whose presence seems associated with the steeper slopes). The relatively unstructured enstrophy throughput then reproduces a more clasical inertial range with lower kurtosis and shallower spectral slope than that of many simulations. However, if the -3 spectrum is brought about by the destruction of coherent vortices by anisotropic Rossby waves we would expect, according to the simulations, anisotropic small scales, whereas observations of atmosphere show isotropic small scale motion (Nastrom and Gage, 1985). It would appear that these issues could be addressed with models of geostrophic turbulence somewhat intermediate between pure turbulence models and atmospheric General Circulation Models.

Energy is also effectively 'input' into the large scale system through convective activity at scales of tens of kilometers. This, while plainly not two-dimensional, occurs at a scale not much smaller than the smallest two-dimensional scales. It has been hypothesized (e.g. Lilly, 1989) that convective input produces a two-dimensional inverse energy cascade, and hence a -5/3 slope. The question arises as to where this energy is dissipated, and what happens when the inverse energy cascade meets the forward enstrophy cascade? Numerical simulations of this phenomena indicate that the enstrophy and energy cascade barely interact (fig.5 and fig. 2e). The intermediate range is made truly invisicid, and an equilibria is reached with a change in spectral slope at an intermediate wavenumber k_c close to the phenomenological estimate

$$k_c \sim \left(\frac{\eta}{\epsilon}\right)^{1/2}$$

where η and ϵ are the energy and enstrophy cascade rates, both taken positive. It seems remarkable that a forward enstrophy cascade, and an inverse energy cascade, can co-exist over the same wavenumber range, with zero dissipation.

This result by no means resolves the issue of the observed change in spectral slope in the atmosphere. It should be regarded merely as evidence that the dual cascade mechanism is a possibility. Other possibilities exist, and should be investigated. These include the possibility that the non-gestrophic energy spectrum of gravity waves, being shallower (possibly a -2 slope) than the geostrophic energy spectrum of the enstrophy range, begins to dominate at smaller scales. Again this idea could be tested with simple models, this time of shallow water turbulence.

6. Discussion. The picture of two-dimensional turbulence which has emerged over the last decade or so, for decaying non-rotating turbulence, is one of the emergence of isolated vortices which, through phase correlations across many length scales, evidently severely curtail the enstrophy cascade. Between the vortices, the enstrophy cascade continues in a business as usual fashion. The vortex dynamics is rather well described by point-vortex like Hamiltonian dynamics, except when vortices collide and enstrophy is dissipated. The total energy spectrum is dominated by the vortices, and, in the slowly evolving period after the vortices have formed but before the system has reduced itself to a very small number of vortices, the energy spectrum tends to be steeper than the classical -3 prediction. If it makes sense to speak of a finite number of vortices, this number decreasing with time, then self-similarity in the field must be foregone (for otherwise there is an infinite number of vortices). This seems to arise because of the 'pac-man' like dynamics of vortices merger, large vortices engulfing the smallest ones as they move through the fluid. Still Benzi *et al.* (1987b) find that the vortex structure is actually self-similar and the situation does not seem entirely clear.

Vortex emergence persists in forced-dissipative calculations, although vorticity kurtosis rarely reaches the extreme values of the decaying calculations. The most basic of the predictions of two-dimensional phenomenology—that of an inverse energy cascade and forward enstrophy cascade—plainly has much truth. In the inverse

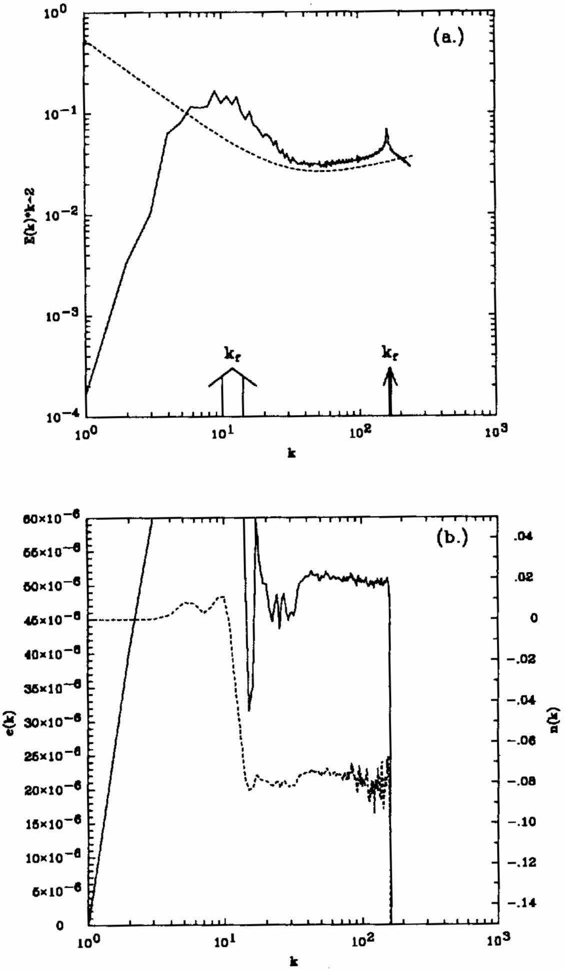


Fig. 5. Flow diagnostics from a 512^2 simulation forced in two distinct wavenumber ranges: $10 \leq k \leq 14$ and $160 \leq k \leq 165$. (a) Time averaged enstrophy spectrum. The dashed line indicates the spectral shape predicted by Leith's (1968) closure following Lilly (1989). (b) Time averaged spectral energy flux to large scales, $\epsilon(k)$ (solid curve), and enstrophy flux to small scales, $\eta(k)$ (dashed curve). (From MV.)

energy cascade coherent structures do not stand out and vortex coalescence appears not to be a dominant mechanism. The predicted -5/3 energy spectrum appears to be fairly well born out, with a Kolmogorov constant between 6 and 10. Experiments done thus far have mainly used a fairly white stochastic forcing, and should be repeated with an instability forcing, or a stochastic forcing with a long time-scale, to ascertain their effects on the energy range. It is rather disquieting that the value of the Kolmogorov constant is not known to greater precision, and that slightly different methods of calculation give slightly different results (see MV), possibly in part due to slight departures from an exact $-5/3$ slope. Warn has suggested (*pers. comm.*) that its value may be non-universal, even though the spectral slope may be. (The argument based on intermittency, is not unique to two-dimensions. See also Kraichnan, 1974.)

In the enstrophy inertial range the classical -3 spectrum appears not to be a robust feature, the primary reason for this being nonlocality of enstrophy transfer. It remains to be seen whether at extremely high Reynolds numbers a -3 spectrum is yet achieved, independent of forcing details. Results at current resolutions indicate that at least 2048^2, and probably much higher, will be required to see qualitative changes from present results. Nevertheless, the notion of an enstrophy transfer to small scales is robust. The enstrophy transfer to small scales occurs largely between vortices, and in this region one may conjecture that KKBL theory has quantitative validity (i.e. -3 slope with a more-or-less universal value of Kolmogorov's constant). Although coherent vortices plainly affect the inertial range dynamics their presence is sensitive to dispersive wave dynamics, in particular to Rossby waves caused by differential rotation. Increasing this is a rather efficient way to reduce the kurtosis and disperse compact structures. Similarly, white noise forcing can scramble the phases and remove the structured nature of the vorticity field.

Inertial range intermittency takes two forms (for the enstrophy range). Most obviously, in decaying turbulence the isolated vortices themselves inhibit the enstrophy cascade and when vortices collide enstrophy is rapidly cascaded to the small scales. More subtly, but perhaps more conventionally, there may be intermittency in the enstrophy flux or dissipation between the vortices, analogous to intermittency in energy dissipation in three dimensional turbulence. There seems little obvious relationship between the two, except that certain arguments suggest that between the vortices the fluid is essentially passive, and therefore the dynamics therein is largely determined by the intense vortices themselves. The effects of dispersive wave dynamics and stochastic forcing on intermittency are not well known, except that both seem likely to reduce it, just as they are destructive influences on isolated vortices.

A couple of dilemmas arose in relation to atmospheric dynamics. Simulations and the phenomenology outlined in this note indicate there is little reason to expect a -3 energy slope in 2D turbulence, or most likely in geostrophic turbulence. Yet this is what is observed in the atmosphere. Why? Further, it is not understood what causes the return to isotropy in the atmosphere. This may related to non-geostrophic effects, or to it having a rather shallow spectrum.

In summary, for pure two-dimensional turbulence, isolated vortices imply the presence of more structure in the vorticity field (in the enstrophy range) than a purely statistical picture seems able to cope with, and the strong non-Gaussianity of the vorticity field probably makes the application of closure theories, including the renormalisation group, suspect. But strong forcing and/or wave-dynamics muddies the waters and restores a neo-classical picture in which structures do not play a dominant role. For all of these reasons a single framework for turbulence—be it structured or statistical—seems unlikely to emerge.

This work was funded by NSF (ATM 8914004) and ONR (N00014-90-J-1618). I am grateful to M. Maltrud for the simulations and illustrations.

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