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Remarks on the predictability properties of two- and three-dimensional flow

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SUMMARY

This paper discusses the relationship between certain well-posedness results for the equations of motion and simple phenomenological arguments pertaining to the predictability of the flow fields. The barotropic and quasi-geostrophic equations have been shown to be well-posed under certain conditions. From this, it has been inferred that Lorenz's conjecture of finite predictability time for any scale of initial error is false. Such an inference is justified if the error energy may be made small by confining the error to small scales of motion. It is shown that this may be achieved, given finite bounds on the vorticity. It is also shown that there is no contradiction between essentially Kolmogorovian phenomenological arguments and the well-posedness of the equations, since the conditions under which a finite predictability time is predicted by the former are those under which the equations do not necessarily admit of smooth solutions. Given smooth enough initial conditions, then phenomenological turbulence arguments and rigorous existence proofs imply that the predictability time of a barotropic fluid may be made as long as we wish if the initial error scale can be made small enough. Both sets of arguments also imply that enstrophy dissipation tends to zero as viscosity tends to zero. For a three-dimensional fluid phenomenology implies a finite predictability time, and non-zero energy dissipation as viscosity tends to zero, implying the ill-posedness of the three-dimensional Euler equations.

1. INTRODUCTION

It has been known for some time that for the equations of motion governing barotropic, or two-dimensional, flow well-behaved solutions exist. That is to say given smooth enough initial conditions the flow does not 'blow-up', even if inviscid. Specifically, in two dimensions in a bounded domain the Euler equations exhibit 'global regularity', meaning the flow remains analytic for all time, provided the initial vorticity is Holder continuous (a condition stronger than ordinary continuity but weaker than differentiability). If the initial velocity is then n-times differentiable (C^n) (it will be at least at once differentiable if vorticity is Holder continuous) it remains so for all times, and in particular solutions with C^∞ (infinitely differentiable) initial conditions remain so always. However, if the initial vorticity is unbounded, or merely discontinuous, the results do not go through (although this is not to say that the flow is then necessarily ill-posed). See Rose and Sulem (1978) for a review. In three dimensions no such results have been obtained, and global regularity may not exist because vortex stretching can lead to *extremely* rapid vorticity growth possibly causing singularities in finite time. In two-dimensional flow the offending term, $\boldsymbol{\omega} \cdot \nabla \mathbf{U}$, is identically zero and vorticity is conserved. In quasi-geostrophic flow vortex stretching arises only through the action of planetary vorticity f and again no catastrophic growth of relative vorticity can occur. Indeed in a series of papers Bennett and Kloeden rigorously demonstrated the important result that the quasi-geostrophic equations are well-posed under fairly weak conditions (see Bennett and Kloeden 1981a, and references therein). Being well-posed in the classical sense means having unique, smooth solutions which depend continuously on the initial data. This has important implications for predictability. Let $\psi_1(t)$ and $\psi_2(t)$ be the streamfunctions of two solutions of the barotropic or quasi-geostrophic equations of motion. Then the theorems imply that at any time $t > 0$ and given any $\varepsilon > 0$, there exists a $\delta(t, \varepsilon) > 0$ such that, if

$$|E\{\psi_1(0) - \psi_2(0)\}| < \delta \quad \text{then} \quad |E\{\psi_1(t) - \psi_2(t)\}| < \varepsilon \tag{1}$$

for any t for which a solution exists. $E(\psi)$ is the energy of the solution and $E(\psi_1 - \psi_2)$ will be referred to as the error energy, or simply the error.

Lorenz (1969) had previously conjectured that the earth's atmosphere might be an example of a fluid system for which for any particular future time there is a limit below which the error energy cannot be reduced, no matter how small the initial error, if not zero. This suggestion seems to stem from phenomenological reasoning and the use of a closure model of two-dimensional turbulence. We shall refer to this as 'finite predictability' as opposed to 'indefinite predictability' implied by (1). Any well-posedness of the equations of motion seems at odds with any such conjecture, as pointed out by Bennett and Kloeden (op. cit.). However, Lorenz's argument does not pertain to point sources of error, but to global disturbances of a characteristic scale, or wavenumber. He had argued that the time taken for an error initially confined to the smallest *scales* of motion to dominate all scales could under certain conditions be finite, no matter how small the scale of the initial error. To make the connection with the regularity proofs it must be shown that we can make the error energy as small as we like by confining the error to ever smaller scales of motion, in the Fourier sense. Although this may seem physically obvious, its proof (given in section 2) is not entirely trivial. We consider only two-dimensional flow, in a doubly-periodic finite domain. It is then shown, in section 3, that the regularity proofs are in fact consistent with simple phenomenological arguments since the latter do imply indefinite predictability for two-dimensional (but not three-dimensional) flow evolving from sufficiently smooth initial conditions. Section 3 also contains a discussion of the zero-viscosity limit of two- and three-dimensional flows. Readers not interested in the mathematical proof may skip section 2.

2. Rigorous predictability properties

The barotropic equation of motion on an f plane is

$$\partial(\nabla^2\psi)/\partial t + J(\psi, \nabla^2\psi) = 0 \tag{2}$$

where

$$J(\psi, \nabla^2\psi) = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi.$$

If $\omega = \nabla^2\psi$, then $d/dt\,(\int\omega^n d\mathbf{x}) = 0$, for $n = 0, 1, 2, \ldots$. Let $S = \int\omega^2\,d\mathbf{x}$, and $E = \int(\nabla\psi)^2\,d\mathbf{x}$. In all cases the integral is over the domain. We suppose the solution of (2) may be written as the double Fourier series

$$\psi(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\psi_{mn}\,\mathrm{e}^{\mathrm{i}(mx+ny)} \tag{3}$$

and to ensure reality of the streamfunction, $\psi_{mn} = \psi^*_{-m-n}$. All lengths are non-dimensionalized by $L/2\pi$ where L is the length of the (square) domain.

Let

$$\hat\psi(x, y) = \sum_{m=-N}^{N}\sum_{n=-N}^{N}\hat\psi_{mn}\,\mathrm{e}^{\mathrm{i}(mx+ny)}$$

and let $\hat\omega = \nabla^2\hat\psi$, etc. We may suppose the 'true' state of the atmosphere is given by $\psi(x, y)$ and our 'observation' of it by $\hat\psi(x, y)$. Then the well-posedness of the dynamical equations will indeed imply the falsehood of Lorenz's conjecture (inasmuch as it pertains only to two-dimensional flow) and hence indefinite predictability if the following theorem is true:

Theorem: At time $t = 0$, given any $\varepsilon > 0$ there exists a finite $N > 0$ and a $\delta > 0$ such that if

$$|\hat\psi_{mn} - \psi_{mn}| < \delta \quad \text{for} \quad |m|, |n| < N$$

then $|E(\psi - \hat\psi)| < \varepsilon$.

In English, the theorem implies that we may make the total error energy as small as we like by accurately measuring all scales of motion down to some observational cut-off scale, provided this scale is small enough. Below we give a couple of proofs, or sufficiency conditions, for the theorem. 'Proof' 1 is given by way of introduction. Note that the theorem need hold only at $t = 0$ (the time at which we make the 'observations').

'Proof' 1. Assume the energy spectrum is a function of some power of the wavenumber $k = (m^2 + n^2)^{1/2}$. Thus suppose $E(k) = Bk^{-n}$ where the total energy is $\int_1^\infty E(k)\,dk$. (Such spectra will be referred to as '$-n$' spectra. The replacement of a discrete spectrum by a continuous one is convenient, and not severe.) Then the energy in the scales $k > N$ is a finite, decreasing function of N if $n > 1$. However, to assume a power law behaviour, for the instantaneous spectrum, is very restrictive. We have really assumed the result, rather than proven it. Note that unless $n > 3$ the enstrophy, and hence the vorticity, are not bounded and the equations no longer have necessarily well-behaved solutions.

Proof 2. This proof assumes ω to be continuous and therefore (by periodicity) bounded—conditions which are also required in the regularity proofs. Differentiability need not be assumed. Note that continuity implies ω and ω^2 are integrable. Enstrophy S is then also bounded by $S \leqslant \omega_{\mathrm{m}}^2 A$, where A is the domain area and ω_{m} is the maximum absolute value of ω within the domain, also denoted $\max|\omega|$.

Define $\phi = \psi - \hat\psi$. Then the energy in the error field is

$$E(\phi) = \int(\nabla\phi)^2\,d\mathbf{x} = \int\nabla\cdot(\phi\,\nabla\phi)\,d\mathbf{x} - \int\phi\,\nabla^2\phi\,d\mathbf{x}.$$

The first integral, which exists, is zero. So

$$\begin{aligned} E(\phi) &= -\int\phi\,\nabla^2\phi\,d\mathbf{x} \\ &\leqslant \int|\phi||\nabla^2\phi|\,d\mathbf{x} \\ &\leqslant \max|\phi|\int|\nabla^2\phi|\,d\mathbf{x} \end{aligned} \tag{4}$$

where $\max|\phi|$ denotes the maximum value achieved by $|\phi|$ within the domain. Now, we can make $|\phi|$ as small as we like everywhere by making the sum of the coefficients in its Fourier expansion sufficiently small. For we have, where a single summation sign implies a sum over all $|m|, |n| < \infty$

$$|\phi| = \left|\sum\phi_{mn}\,\mathrm{e}^{\mathrm{i}(mx+ny)}\right| \leqslant \sum\left|\phi_{mn}\,\mathrm{e}^{\mathrm{i}(mx+ny)}\right| = \sum|\phi_{mn}|. \tag{5}$$

Further, we can make the infinite sum on the r.h.s. of (5) as small as we please, by setting $|\hat\psi_{mn} - \psi_{mn}| < \delta(N)$ for all $|m|$, $|n| < N$, no matter what the value of the remaining coefficients. To see this write

$$\sum|\phi_{mn}| = \sum_{<}|\phi_{mn}| + \sum_{>}|\phi_{mn}| \tag{6}$$

where the sums written $\Sigma_>(\Sigma_<)$ denote partial sums over all $|m|$, $|n|$ greater than (less than) N.

Using the Cauchy–Schwarz inequality, and the fact that the individual terms in the first sum on the r.h.s. are all less than δ, (6) becomes

$$\sum |\phi_{mn}| \leq (2N+1)^2\delta + \left| \left\{ \sum_{>} (m^2+n^2)^2 |\psi_{mn}|^2 \right\}^{1/2} \times \left\{ \sum_{>} (m^2+n^2)^{-2} \right\}^{1/2} \right|$$

$$\leq (2N+1)^2\delta + S^{1/2}D^{1/2}N^{-1}.$$

Hence, using (5),

$$|\phi| \leq (2N+1)^2\delta + S^{1/2}D^{1/2}N^{-1}. \tag{7}$$

We have written $\Sigma_>(m^2+n^2)^{-2} \sim DN^{-2}$, where D is a finite constant. This is valid for large N, as can be seen easily by approximating the sum by the integral $\int_N^\infty (2\pi k\, dk)/k^4$. (Any additional terms in the summation are all of higher order in N^{-1}.) By choosing δ small enough (such that $N^2\delta \to 0$ as $N \to \infty$) and N large enough we may make ϕ as small as we like everywhere. To complete the proof requires that the integral

$$I = \int |(\omega - \hat{\omega})|\, d\mathbf{x} = \int |\nabla^2\phi|\, d\mathbf{x}$$

is bounded for all N. Certainly

$$I \leq A\,(\max|\omega| + \max|\hat{\omega}|) \tag{8}$$

which is bounded for finite N. As $N \to \infty$, $\hat{\omega}$ is not, however, obviously bounded since there exist continuous, periodic functions whose Fourier series do not converge everywhere. However, we have

$$I \leq \int |\omega|\, d\mathbf{x} + \int |\hat{\omega}|\, d\mathbf{x}. \tag{9}$$

The first term is bounded by assumption (its upper bound is $A \max|\omega|$). The second term is also bounded because, by the Cauchy–Schwarz inequality

$$\int |\hat{\omega}|\, d\mathbf{x} \leq \left(\int \hat{\omega}^2\, d\mathbf{x} \right)^{1/2} A^{1/2}.$$

But even as $N \to \infty$ the right-hand side is certainly bounded by Bessels inequality, which ensures

$$\lim_{N\to\infty} \int \hat{\omega}^2\, d\mathbf{x} \leq S. \tag{10}$$

Note that we have not proved or required that $\hat{\omega} \to \omega$ as $N \to \infty$. To do so requires stronger assumptions about ω.

Using (4), (7), (9) and (10) the theorem follows, namely that we can make the energy in the error (ϕ) field as small as we like by making the first N Fourier components of its streamfunction sufficiently small.

3. Phenomenological predictability properties

We have shown that, given bounded vorticity, the energy in the initial 'error field' may be made as small as we like by choosing N sufficiently large, where N is the wavenumber beyond which no information exists regarding the flow. A corollary is that

the series (3) is uniformly convergent. Although the result is perhaps physically obvious, it seems worthwhile cementing the connection between spectral predictability theories *à la* Lorenz (1969) and regularity proofs of the simplified equations of motion. The simple consequence is that the time taken for an error, concentrated initially at small scales, to dominate large scales may be made as long as we like by choosing the initial error scale small enough, provided the conditions for well-posedness are satisfied.

Arguments to the contrary (Lorenz 1969; Lilly 1972) may be expressed as follows. (This and all of the arguments following are heuristic.) An eddy turnover time τ_k may be defined, partly on dimensional considerations, by

$$\tau_k = \{k^{-3}/E(k)\}^{1/2}. \tag{11}$$

If $E(k) = Ck^{-n}$, then $\tau_k = C'k^{(n-3)/2}$ where C and C' are constants. This is the time taken for flow structures of scale $1/k$ to be distorted by a velocity $\{kE(k)\}^{1/2}$. Now suppose – and this is the crucial assumption – that the time taken for error to propagate from a scale k to scale $k/2$ is proportional to τ_k. Then the total time for error to propagate through scales from $k = k_1$ to $k = 1$ is

$$T = \sum_{m=0}^{p} C' 2^{m(n-3)/2} \quad (\text{where } 2^p = k_1)$$

$$= C' \left\{ \frac{1 - 2^{(p+1)(n-3)/2}}{1 - 2^{(n-3)/2}} \right\}. \tag{12a}$$

Alternatively, in a continuous wavenumber spectrum

$$T = \int_1^{k_1} \frac{d(\ln k)}{\sqrt{\{k^3E(k)\}}} = [C'k^{(n-3)/2}]_1^{k_1} \{2/(n-3)\}. \tag{12b}$$

As $k_1 \to \infty$ both estimates converge for $n < 3$, and so predictability is always finite, in apparent contradiction to the well-posedness results. Note that a finite predictability time T need only be the time taken for error to *contaminate* all scales (and not necessarily dominate all scales) for ill-posedness to ensue.

However, a spectrum shallower than k^{-3} is not expected for two-dimensional flow. Dimensional arguments suggest that one write, for inviscid flow,

$$\eta_k = k^3E(k)/\tau_k \tag{13}$$

where η_k is the enstrophy cascade rate. If (11) is used for τ_k, and we demand η_k to be independent of k, we find

$$E(k) = K_0\eta^{2/3}k^{-3} \tag{14}$$

where K_0 is an order-one constant. If the flow is inviscid, and (14) is valid for all wavenumbers, then predictability is indefinite. The regularity proofs are not violated. However, the dimensional arguments are not set in stone. (See, for example, Frisch *et al.* (1978) for alternative models.) Suppose a spectrum shallower than $n = 3$ exists, then the finite predictability apparently violates the well-posedness. However, for such a shallow spectrum the total enstrophy is infinite, which in turn implies that the vorticity somewhere in the fluid is infinite. Now, enstrophy is conserved by the fluid (indeed it is this which enables one to construct existence theorems for the flow for both barotropic and quasi-geostrophic motion). So that unless enstrophy is initially unbounded, it remains bounded. But if it is initially unbounded then so, somewhere, is vorticity; the regularity proofs fail and in particular condition (1) is not necessarily valid. Again then, there is no contradiction between the phenomenological arguments leading to (12) and the well-posedness proofs.

Kraichnan (1971) argues that for the -3 spectrum enstrophy transfer is insufficiently local for (14) to be consistent with a constant η_k. Instead of an eddy-turnover time given by (11), he suggests

$$\tau_k = \left\{\int_1^k p^2 E(p)\,dp\right\}^{-1/2}. \tag{15}$$

For any spectrum steeper than -3, this is to be preferred over (11) since the latter gives $\tau_k \to \infty$ as $k \to \infty$, which is unphysical. For $n < 3$, the two are equivalent for large k. Use of (15) in (13) leads to the logarithmically corrected spectrum:

$$E(k) = K_0 \eta^{2/3} k^{-3} \{\ln(k/k_2)\}^{-1/3} \tag{16}$$

where k_2 is some lower cut-off wavenumber. The predictability time of a fluid with such a spectrum is then given by

$$T = \int_1^{k_1} \frac{d(\ln k)}{\{\int_1^k p^2 E(p)\,dp\}^{1/2}}$$
$$\propto \{\ln(k_1/k_2)\}^{2/3} \quad \text{for } k_1 \gg k_2. \tag{17}$$

This also diverges as $k_1 \to \infty$, implying indefinite predictability.

Thus, for inertial ranges of infinite extent the phenomenology predicts indefinite predictability for both a -3 and a logarithmically corrected -3 spectrum. These are actually stronger (but less well-founded) results than can be obtained rigorously, since for both spectra the enstrophy $\int_1^\infty k^2 E(k)\,dk$ is infinite and the proofs can no longer guarantee that (1) be satisfied.

If enstropy is initially bounded, its value remains constant. If initial conditions with $\nabla\omega$ finite are imposed, then $\nabla\omega$ will remain finite (but may grow very rapidly). Now, $\nabla\omega$ will be infinite if the energy spectrum is shallower than $n = 5$ as $k \to \infty$. Thus it must take an infinite time to set up a putative inertial range shallower than $n = 5$. Loss of predictability then never arises. If the initial conditions are C^∞, say because all the energy is confined to Fourier modes of finite k, the spectrum must fall off at least exponentially as $k \to \infty$ for all time. These conditions have interesting consequences for the formal zero viscosity limit of the equations. First, though, let us note that if viscosity is non-zero the right-hand side of (2) is replaced by $\nu\nabla^4\psi$. Since a positive viscosity can only act to decrease enstrophy a spectrum shallower than k^{-3} can arise only if enstrophy is initially infinite as before. So viscosity certainly does not lead to loss of predictability.

Viscosity can be expected to become important at scales smaller than k_ν^{-1}, obtained by equating an energetic timescale (15) with a viscous timescale $1/\nu k^2$. For $k_\nu \gg 1$ this gives, assuming $E(k) = K_0 \eta^{2/3} k^{-n}$,

$$k_\nu \sim (\eta^{2/3}/\nu^2)^{1/(n+1)} \quad \text{for } n < 3 \tag{18a}$$

or

$$k_\nu \sim \eta^{1/6} \nu^{-1/2} \quad \text{for } n > 3 \tag{18b}$$

while for the log-corrected -3 spectrum we find

$$\ln(k_\nu/k_2)^{-1/3} k_\nu^2 = \eta^{1/3} \nu^{-1}. \tag{18c}$$

These expressions are valid after the inertial range has been set up (which may require infinite time as $\nu \to 0$). Beyond k_ν the energy spectrum will fall off rapidly (steeper than

k^{-5} or else enstrophy dissipation becomes infinite). The inertial range itself ($k < k_\nu$) may in principle be made as large as we wish by reducing the coefficient of viscosity, or equivalently increasing the Reynolds number (Re). The total enstrophy dissipation as $\nu \to 0$ is given by

$$\dot{S} = \lim_{\nu \to 0} \nu \int_1^{k_\nu} k^4 E(k)\,dk. \tag{19}$$

This converges to zero for a spectrum steeper than $n = 3$, but it is independent of ν for an $n = 3$ or a logarithmically corrected -3 spectrum (the integral then converges to $\sim\eta$). However, this is not to say enstrophy dissipation remains finite if viscosity is turned off. For at such time, the flow has bounded enstrophy. Hence the asymptotic spectrum (i.e. as $k \to \infty$) must always be steeper than $n = 3$ and the integral (19) remains zero for all time. Different arguments with a similar conclusion are given in Bennett and Kloeden (1981b).

Phenomenological arguments lead to the same conclusion. One may suppose that, after viscosity is 'turned off' the high wavenumber limit of a -3 (or log-corrected -3) inertial range spreads to higher and higher wavenumbers. By analogy with the arguments leading to (12) one may suppose that the time it takes to spread from wavenumbers k to $2k$ is proportional to the eddy turnover time τ_k. Thus it takes an infinite time for such an inertial range to cover all wavenumbers, and until such time (19) is zero. Closure arguments (which many would say are equally phenomenological) by Pouquet *et al.* (1975) also imply zero enstrophy dissipation in the zero viscosity limit.

It is instructive to carry through the equivalent phenomenological arguments for a three-dimensional fluid, for which no global regularity proof has been obtained. For such a fluid, Kolmogorov dimensional arguments suggest an inertial range of the form

$$E(k) = K_0' \beta^{2/3} k^{-n}$$

where $n = 5/3$, β is the energy cascade rate and K_0' an order-one constant. The energetic eddy turnover times $\{k^3 E(k)\}^{-1/2}$ and $\{\int_1^k p^2 E(p)\,dp\}^{-1/2}$ are now roughly equivalent, because most of the contribution to the integral in the latter expression comes from $p \sim k$. The time taken for errors initially confined to scales of wavenumber greater than k_1 to spread to wavenumber $k_2 (< k_1)$ is

$$T = \int_{k_2}^{k_1} \frac{d(\ln k)}{\sqrt{\{k^3 E(k)\}}} \sim \beta^{-1/3}(k_2^{-2/3} - k_1^{-2/3})$$

which evidently converges for large k_1. Thus, no matter how small the initial error (provided it lies in the inertial range), scales of characteristic size l_0 become unpredictable in a detailed sense after a time $\beta^{-1/3} l_0^{2/3} \sim E(l_0)^{-1/2} l_0^{3/2} \sim l_0/|U(l_0)|$.

The zero viscosity limit of three-dimensional flow also differs from the two-dimensional case. Equating the energetic turnover time to the viscous timescale indicates viscosity will be important for wavenumbers greater than order k_ν, where $k_\nu = [\beta^{1/3}\nu^{-1}]^{2/(n+1)}$. The energy dissipation in the inviscid limit is then given by

$$\dot{E} = \lim_{\nu \to 0} \nu \int_1^{k_\nu} k^2 E(k)\,dk.$$

This is independent of ν for $n = 5/3$, zero for a steeper spectrum. However, in contrast to the two-dimensional case, there is no conservation law preventing such a range extending infinitely (energy conservation is a very weak constraint on the allowable power spectrum). Further, the phenomenology now does imply that a $-5/3$ inertial range will extend to infinity within a finite time (T_*) using (12) (with a minor correction due to the overall amplitude of the energy spectra falling as it extends, to conserve energy). For $t < T_*$, energy dissipation is zero. For $t > T_*$, energy dissipation is finite. If the initial spectrum is concentrated at scales l_0 and with r.m.s. velocity $|u^2(0)|^{1/2}$, the 'catastrophe' will occur in a time (T_*) of order $l_0/|u^2(0)|^{1/2}$. Such arguments suggest that the three-dimensional inviscid Navier–Stokes equations are not well-posed.

4. Conclusions

Given certain smoothness conditions (which imply boundedness) on the vorticity of a two-dimensional fluid, both heuristic arguments based on error transfer being related to eddy turnover time and rigorous results pertaining to the existence of solutions of the equations of motion imply that the predictability time of the fluid may be made indefinitely long by choosing the scale of the initial error small enough. (There is no a priori reason why phenomenology and regularity should lead to the same results, although they should be consistent unless one (presumably phenomenology) is wrong. The rigorous arguments do not necessarily demand *ill*-posedness if vorticity is not bounded, rather they guarantee well-posedness if certain conditions (like Holder-continuity) are met. They give bounds which must be satisfied by any heuristic theory but can only falsify it if such a theory predicts ill-posedness when well-posedness is known.)

The rigorous results on enstrophy dissipation in the zero-viscosity limit (that it goes to zero) depend (as do all the rigorous proofs) on the imposition of smooth initial conditions. Again the phenomenology is in accord. Heuristically one sometimes imagines an inertial range stretching to infinity as the Reynolds number increases. It is precisely this which the proofs prohibit, unless one unrealistically (and experimentally unverifiably) takes the infinite time limit before letting viscosity go to zero in the expression, (18), governing the high wavenumber cut-off of the inertial range. However, for any non-zero viscosity one still does imagine that enstrophy is dissipated faster than energy, and selective-decay hypotheses (e.g. Bretherton and Haidvogel 1976) which conjecture that a fluid will choose to be in a minimum enstrophy state for a given total energy are not, at least by this result, invalidated. Enstrophy, in any case (even zero viscosity), will always be expelled from the large-scale.

It is interesting that phenomenology predicts ill-posedness (and finite enstrophy dissipation) for spectra shallower than -3, and that spectra strictly steeper than this (as $k \to \infty$) are required for rigorous well-posedness. If one believes the phenomenology, then this suggests that a little (but only a little) improvement can be made in the rigorous proofs for two-dimensional flow. For a -3 (perhaps log-corrected) spectrum extending to infinity enstrophy is unbounded; the phenomenology still predicts well-posedness whereas the regularity proofs can say nothing. For a three-dimensional fluid, the lack of a global regularity proof for the Euler equations is consistent with the phenomenological prediction of a finite predictability time for any initial error within the inertial range, and the ability of a $-5/3$ inertial range to extend infinitely (thereby implying finite energy dissipation) as $\nu \to 0$. Either the 3-D Euler equations are ill-posed, or the phenomenology is incorrect as $Re \to \infty$. Assuming the former, boundary layer and other three-dimensional processes ultimately provide the theoretical predictability limits in the atmosphere. There is no point in ever having an observing system of much finer resolution than the scales at which such processes become important, because error growth is so rapid within them.

Our present observational network is probably sufficiently coarse that such problems are unlikely to be relevant.

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