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Instability and Flow Over Topography

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The stability of quasi-geostrophic β-plane flow over topography is examined. The approach is to first calculate the stationary, asymmetric response to a uniform zonal current flowing over topography, and then calculate the stability properties of the total, zonally asymmetric, field. Under many circumstances this flow is unstable, barotropically and/or baroclinically particularly if the asymmetric flow is of large amplitude. This is demonstrated first using a long-wave approximation, which examines the stability with respect to perturbations of large meridional scale. So-called form-drag instability then ensues. This may be thought of as a special, nonlocal, form of isosceles triad interaction involving the zonal flow interacting with the topography and another "free" mode of topographic scale. For topography consisting of a single Fourier mode, instability then arises only if the zonal current is eastward and exceeds that required for resonance. However, in general other triads exist in which the asymmetric flow, if its amplitude is large enough, is always unstable, for any value of the zonal current. In particular, flow with the zonal current slightly below the resonant value can be unstable. This implies that resonantly amplified stationary waves, sometimes cited as possible mechanisms for blocks, will decay rapidly through their interaction with other modes, unless further nonlinear equilibration occurs. Certain integral constraints prove useful in ascertaining necessary conditions for instability, both for topographic instability and the zonally symmetric (but continuously stratified) baroclinic instability problem.

1. INTRODUCTION

Two problems in atmospheric dynamics are to understand the amplitude and structure of zonally averaged flow and the time-averaged asymmetric flow. The separation is arbitrary but convenient. There is no entirely satisfactory theory which explains well the essential features of either of these phenomena. Even *given* the zonal flow, there are few clear ideas regarding the amplitude of the time-mean asymmetric flow. If the amplitudes of the asymmetric features are sufficiently small, linear theory expanded about the zonal mean state will be a good approximation. If not, stationary nonlinearities will affect the amplitude predicted by linear theory. Further, the stationary asymmetric flow may be unstable, spawning transient eddies which will extract energy from it.

The occasional occurrence of large asymmetric anomalies in the flow (e.g. 'blocks') also lacks a theory, (or perhaps suffers from too many). The idea common to many, for example Charney and deVore (1979) and Tung and Lindzen (1979), is that a particular component of the flow may be preferentially excited by the resonant amplification of a topographic wave, through the interaction of the zonal mean flow and the topography. Resonant amplification occurs at the scale for which the phase speed of a Rossby wave is zero. In the theory of Tung and Lindzen (1979) as the zonal flow, for some reason, passes through a resonant value selective preferential amplification occurs, which may have the shape or structure of blocks. In the theory of Charney and deVore (1979) and Hart (1979) the resonantly excited wave is able to transfer sufficient heat or momentum to balance the forcing on the mean flow, and a stationary equilibrium results. Aside from the theory of Hart, which is valid only for highly anisotropic flow of large meridional scale, such theories are valid only if a severely truncated set of Fourier modes describes the flow well and the asymmetric flow is the *linear* response to the zonal current. Davey (1980) discusses some possible effects of nonlinearity and shows that they appear *not* to be small near resonance. Although in the above cited papers one or more of the stationary equilibria was found to be stable given the model truncation, the stability of the flow in the presence of other modes was not examined. If the flow is then found to be unstable, and the growth rate is sufficiently large, such a stationary equilibrium will not necessarily persist and the "block" will decay.

The stability of stationary, linear solutions (and the exact inviscid nonlinear solutions which have precisely the same form as the inviscid linear solutions) clearly has implications for the time mean asymmetric flow also. For if the asymmetric flow predicted by simple stationary theory is unstable then the actual time averaged asymmetric flow will presumably have an amplitude *smaller* than that given by the stationary theory, since we would expect energy to be transferred from the stationary flow to the transient flow. If the flow is stable, on the other hand, the transient waves which exist in the atmosphere would interact with the stationary flow with little or no phase coherence. They would essentially be a stochastc forcing, and have little effect on the amplitude or phase of the stationary flow, except perhaps in moving it from one stationary equilibrium point to another.

The stability of large scale flow over topography is therefore of paramount importance for both the time-mean flow and blocking flow. This paper is concerned with that problem. I shall not be concerned especially with the value of the initial zonal mean flow, generally taking it to be given. For tractability, and in order to isolate essential physical mechanisms, most of the analysis concerns one and two-layer flow on a β-plane. The idea of "topographic instability" is not new, if by topographic instability one means the destabilisation of a zonal current stable in the absence of topography. A clear example of the destabilizing effects is presented by deSzoeke (1983) in his analysis of the effects of a wavy bottom boundary on the Eady problem. He demonstrated baroclinic instability of a zonal current stable in the presence of a smooth lower boundary. Charney and Flierl (1981) give an example of two-layer zonal flow destabilized by topography. In general a purely zonal flow over topography is not, however, a solution to the equations of motion, unless the zonal flow vanishes at the surface. Hence the approach presented below is somewhat different, in that we first calculate the stationary response set up by the interaction of a given uniform zonal current and the topography. The instability of the complete field (uniform zonal current plus zonally asymmetric response) is then examined. The instability then results from a triad-interaction involving the stationary asymmetric flow and two free modes. In deSzoeke's case the topography itself acts as intermediary between the two free modes. Such instabilities, if important, would affect and perhaps destroy the equilibrium solutions of truncated-spectral-expansion models of flow over topography (e.g. Charney and deVore, 1979) and their baroclinic descendents (e.g. Roads, 1981). These models typically contain only three modes, representing the zonal flow and two modes of the same scale as the topography. Such a truncation allows form-drag instability (Section 4) but other triad-instabilities are not allowed. In such models stationary equilibria exist close to resonance, and flow with the zonal wind just below the resonance value (essentially $\bar{u}<\beta/k^2$) is stable. In Hart's model, which is not based on a truncated spectral expansion, only forced flow components with little meridional variation are allowed, and again only super-resonant instability exists. In both cases the presence of other modes, allowing subresonant topographic instability, may therefore qualitatively alter the picture.

It is the aim of this paper to illustrate the physical basis of topographic instability in a simple and explicit form. Both barotropic and baroclinic instability will be found. The instability mechanisms occurring near resonance have been examined by Pedlosky (1981). Sasamori and Youngblut (1981) and Neelin and Lin (1984) have considered, numerically, the stability problem of forced stationary waves and Gill (1974) and others have addressed the problem of the stability of free waves. The analysis presented below extends these and, it is hoped, clarifies and unifies the basic mechanisms involved. In particular the analysis of Section 4 requires no *ad hoc* spectral truncation and the analysis of Section 5 does not assume perturbations of similar scale to the topography. Further, the similarities between barotropic and baroclinic instabilities for both the topographic and Rossby wave stability problem are brought out, and there is some use of integral constraints in demonstrating sufficient conditions for nonlinear stability.

In Section 2 a scale analysis of the equations is performed. Section 3 is concerned with the exact nonlinear, inviscid solutions for flow over large-scale topography. In Section 4 we consider a "long wave approximation", which leads to an asymptotic description of form-drag instability. Section 5 contains a slightly more general, but less well-founded, stability analysis in part resembling Gill's (1974) analysis of the stability of a Rossby wave. In addition necessary conditions for baroclinic instability are derived. Section 6 contains a summary and conclusions.

2. SCALE ANALYSIS OF BAROTROPIC FLOW OVER TOPOGRAPHY

2a. Relationship to multiple equilibrium theories

The barotropic vorticity equation expresses the conservation of the potential vorticity of the flow, except for effects of friction and forcing. It may be written

$$\partial Q/\partial t+(\Psi, Q)=-\nu\nabla^2\Psi, \tag{2.1}$$

where

$$Q=q+\beta y, \qquad q=\nabla^2\psi+h.$$

It is useful to separate out a constant zonal flow by writing

$$\Psi=-Uy+\psi,$$

whence (2.1) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}+U\frac{\partial}{\partial x}\nabla^2\psi+J(\psi,\nabla^2\psi+h)=-U\frac{\partial h}{\partial x}-\nu\nabla^2\psi. \tag{2.2}$$

The parameter ν is a coefficient of friction, the process represented by $\nu\nabla^2\psi$ being surface drag, or Ekman pumping. h is proportional to the surface topographic height and is related to the dimensional height h^* by $h=fh^*/H$ where H is the depth of the fluid. Thus h has units of $(time)^{-1}$. All other notation in (2.2) is standard.

We shall first consider the circumstances under which a linear approximation to (2.2), namely

$$\partial(\nabla^2\psi)/\partial t+\beta\partial\psi/\partial x+U\partial(\nabla^2\psi)/\partial x=-U\partial h/\partial x-\nu\nabla^2\psi, \tag{2.3}$$

is valid. Before proceeding with the scale analysis, note that the system may be made precisely equivalent to that used in the quasi-linear barotropic multiple-equilibrium theories by writing down an equation governing the evolution of the mean field U. If the energy of the system is defined by

$$E=\tfrac{1}{2}\int[U^2+(\nabla\psi)^2]\,d\mathbf{x},$$

where the integral is over the entire domain, then conservation of energy [for the unforced, inviscid case $(r\to 0, \nu\to 0)$] is achieved by writing

$$\partial U/\partial t+A^{-1}\int\psi(\partial h/\partial x)\,d\mathbf{x}=-r(U-U_0). \tag{2.4}$$

A is the domain area, and the right-hand side represents a Newtonian type momentum forcing. Such an equation may also be derived from a more or less *ad hoc* fashion from the geostrophic momentum equation. Equations (2.2) and (2.4) form a closed coupled set of equations for the evolution of ψ and U. The simplest system to consider is then obtained by allowing only one wave to exist and interact with the mean flow. Thus $\psi = \psi_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x}) + \text{c.c.}$, $h = h_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x}) + \text{c.c.}$, where c.c. denotes complex conjugate. The Jacobian term in (2.2) vanishes, and (2.3) and (2.4) become

$$-k^2 \partial\psi_{\mathbf{k}}/\partial t + ik_x \beta \psi_{\mathbf{k}} - Uik_x k^2 \psi_{\mathbf{k}} = -Uik_x h_{\mathbf{k}} + \nu k^2 \psi_{\mathbf{k}} \tag{2.5}$$

and

$$dU/dt - \mathrm{Re}(\psi_{\mathbf{k}} h_{\mathbf{k}}^* ik_x) = -r(U - U_0) \tag{2.6}$$

Writing $\psi_{\mathbf{k}} = a + ib$, and $h_{\mathbf{k}} = h_r + ih_i$,

$$da/dt + Cb = B_1 U - \nu a, \quad db/dt - Ca = B_2 U - \nu b, \tag{2.7a, b}$$

$$dU/dt - (h_i a - h_r b)k_x = -r(U - U^*), \tag{2.7c}$$

where $C = k_x(\beta - Uk^2)/k^2$, $B_1 = -Uk_x h_i/k^2$ and $B_2 = Uk_x h_r/k^2$. These are identical in structure to the equation of Charney and deVore, and Hart. Such equations are clearly only valid when the Jacobian term $J(\psi, \nabla^2\psi + h)$ (representing wave–wave interactions) is small. Although the system is nonlinear (because of wave–meanflow interactions), if the zonally averaged field (U) is known the zonally asymmetric field is simply the linear response to this.

2b. Scale analysis

Let L be a characteristic length scale and k be a characteristic wavenumber such that $k = 1/L$. The familiar non-dimensional wavenumber k' is then obtained by the relationship $k' = k(L_D/2\pi)$, where L_D is the domain length. Thus $\partial/\partial x \sim L^{-1} \sim 2\pi k'/L_D$. It is convenient to consider three wavenumber regimes

$$\text{i)}\ \ Uk^2 > \beta, \qquad \text{ii)}\ \ Uk^2 < \beta, \qquad \text{iii)}\ \ Uk^2 \approx \beta.$$

In case (i), linear theory is applicable if there is a balance between vorticity advection and topographic forcing, i.e. when

$$U\partial(\nabla^2\psi)/\partial x \sim U\partial h/\partial x,$$

implying

$$\psi \sim h/k^2.$$

The topographically induced nonlinear terms $[J(\psi, \nabla^2\psi + h)]$ are then $O(h^2)$. Thus, nonlinearity is small when the parameter ε_u is small, i.e.

$$\varepsilon_u \ll 1,$$

where

$$\varepsilon_u = h/Uk = hL/U. \tag{2.8}$$

As the length scale gets *smaller*, nonlinearity becomes *less* important. Let $U \sim 10\,\mathrm{m/s}$, $\beta = 1.5\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$. Then the largest scale we need consider in case (i) occurs when $Uk^2 \sim \beta$, giving a length scale of 816 km (for a wavelength, multiply by 2π). For a channel length of 20,000 km, this corresponds to a nondimensional wavenumber 3.9. To obtain a value of ε_u of unity then requires a mountain height of $h \approx 1.23\,10^{-5}\,\mathrm{s}^{-1}$, or a dimensional height of about 1.23 km. Since spectral components of topography in the atmosphere typically have components of order a few hundred meters, condition (2.8) is not especially well satisfied but nor is it strongly violated.

At planetary scales [case (ii)] linear theory requires the balance:

$$\beta\partial\psi/\partial x \sim U\partial h/\partial x,$$

implying

$$\psi \sim Uh/\beta.$$

The nonlinear term $J(\psi, \nabla^2\psi + h)$ is dominated by $J(\psi, h)$, since $J(\psi, h)/J(\psi, \nabla^2\psi) \sim \beta/Uk^2$, and $\beta \gg Uk^2$ by assumption. "Nonlinear" terms [i.e. $J(\psi, h)$, which although linear in ψ, is generally ignored in linear analyses] are indeed unimportant when

$$\varepsilon_\beta = kh/\beta \ll 1. \tag{2.9}$$

At the transition wavenumber $k_\beta = (\beta/U)^{1/2}$ we of course again have $\varepsilon_\beta = 1$ when $h \approx 1.23$ km. At larger scales condition (2.9) is better satisfied because k gets smaller. However the spectral components of the topography become correspondingly larger, and may be as large as several hundred meters. Again then for moderately sized topography (say a few hundred metres) nonlinearity may be small but is not negligible.

In case (iii) there is cancellation between mean advection of vorticity and the advection of planetary vorticity. If U is allowed to evolve (2.4) [or (2.7)], frictional effects become important in determining how close the system is to resonance. Close to resonance let

$$\Delta = (\beta - Uk^2),$$

where $(\Delta/\beta) \ll 1$. Equation (2.3) becomes

$$\partial(\nabla^2\psi)/\partial t + \Delta \partial\psi/\partial x = -U\partial h/\partial x - \nu\nabla^2\psi. \tag{2.10}$$

Still neglecting friction, we see that $\psi \approx Uh/\Delta$. For weak nonlinearity we require the parameter $\varepsilon_r = ha/\Delta$ to be small, where a is the resonant wavenumber $(\beta/U)^{1/2}$, and we have assumed that the nonlinear interactions around resonance are fairly local in k-space. The smallness of ε_r is clearly quite a stringent condition for small Δ, not generally satisfied. The magnitude of Δ may be calculated easily enough using (2.7), [or (2.5) and (2.6)] supposing them for the moment to be valid. From (2.5) the solution for ψ is

$$\psi_{\mathbf{k}} = Uik_x h_{\mathbf{k}}/(\nu k^2 - ik_x\Delta). \tag{2.11}$$

The form-drag $D(U)$ on U is then

$$D(U) = Uk_x^2|h_{\mathbf{k}}|^2 \nu k^2/(\nu^2 k^4 + k_x^2\Delta^2). \tag{2.12}$$

In a steady state this is balanced by the momentum forcing, so that

$$r(U_0 - U) = Uk_x^2|h_{\mathbf{k}}|^2 \nu k^2/(\nu^2 k^4 + k_x^2 \Delta^2). \tag{2.13}$$

Note that the form-drag is zero if the flow is inviscid, unless the denominator of (2.12) also vanishes. Considering only the case of small friction we have

$$\Delta^2 \approx \nu U|h_{\mathbf{k}}|^2 k^2/r(U - U_0). \tag{2.14}$$

Note that (2.14) is consistent with (2.13) since $\nu^2 k^4$ always tends to zero faster than $k_x^2 \Delta^2$ as $\nu \to 0$. Thus, as $\nu \to 0$ (2.13) has three solutions, one at $U = U_0$ when both left and right-hand sides of (2.13) are zero, and two close to (and on either side of) $U = \beta/a^2$ where the form-drag is $-r(\beta/a^2 - U_0)$. These latter two are the resonant solutions. The distance from resonance is then governed by (2.14) with U and k taking the resonant values β/a^2 and a on the right-hand side. As friction gets smaller the wave amplitude (in the resonant solution) gets larger. The nonlinearity parameter $\varepsilon_r = ha/\Delta$ tends toward $r(U_0 - U)/\nu U$, i.e. it becomes larger, and any *a priori* elimination of nonlinearity is inconsistent, whatever the value of the surface topography. Note that the solution at resonance must be unstable. For if we increase the zonal wind slightly the form-drag becomes zero, there is nothing to balance the momentum forcing so the solution tends toward the forcing equilibrium.

To summarize, then, we have demonstrated that, while for sufficiently small mountains we perhaps could ignore nonlinearity away from resonance, it is not permissible to do so near a resonance if the zonal flow is determined by form-drag and friction is small. This is essentially consistent with Davey's (1980) findings. (We have considered only nonlinearity induced by the topography, ignoring transient eddies due to those flow instabilities which are essentially independent of topography.) Of course, nonlinearity under some circumstances may be unimportant because $J(\psi, q) = 0$, even though the terms in the Jacobian $(\partial\psi/\partial x \, \partial q/\partial y)$ are individually large. This is the case in some exact (normally inviscid) solutions of flow over topography, as described below. However, the above analysis suggests that unless these solutions are stable, the instability will be large enough to destroy the form of the solution. Also, such nonlinearities should clearly not be eliminated *a priori*.

3. EXACT NONLINEAR SOLUTIONS FOR ONE AND TWO LAYER FLOWS

In this section nonlinear, steady exact solutions for one and two layer quasi-geostrophic flow over topography are derived. The solutions are unique provided there is no friction, no recirculation regions and given a constant upstream flow. The resonant conditions for a few special cases are examined. This section is a preliminary to the stability studies of Sections 4 and 5.

3a. Barotropic flow

Steady inviscid solutions to (2.1) may be written

$$Q=Q(\Psi)$$

since then $J(\Psi,Q)=0$. Given the upstream (or downstream) boundary condition, far from any topographic influence

$$Q=\beta y, \quad \Psi=-Uy, \tag{3.1}$$

where U is a constant the general solution is

$$Q=-\beta\Psi/U, \tag{3.2}$$

provided no closed contours exist in the flow. Note that writing $Q=q+\beta y$ and $\Psi=-Uy+\psi$, we have still $q=-\beta\psi/U$. Thus both the mean fields and the eddy (zonally asymmetric) fields satisfy the condition (3.2), and so a stream function ψ given by $(\nabla^2\psi+h)=-\beta\psi/U$ is a solution of

$$\frac{\partial}{\partial t}\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}+U\frac{\partial}{\partial x}\nabla^2\psi+J(\psi,\nabla^2\psi+h)=-U\frac{\partial h}{\partial x}. \tag{3.3}$$

In a zonally re-entrant channel, where no upstream boundary condition like (3.1) exists, (3.2) is still a solution of (3.3), but not necessarily the only one.

3b. Two layer flow

Exact solutions may also be written down for two layer flow over topography. The potential vorticity equations may be written, at the upper and lower levels, as

$$\frac{\partial q_1}{\partial t}+\beta\frac{\partial\psi_1}{\partial x}+U_1\frac{\partial}{\partial x}\nabla^2\psi_1+\frac{\lambda^2}{2}\left(U_1\frac{\partial\psi_2}{\partial x}-U_2\frac{\partial\psi_1}{\partial x}\right)+J(\psi_1,q_1)=0,$$

$$\frac{\partial q_2}{\partial t}+\beta\frac{\partial\psi_2}{\partial x}+U_2\frac{\partial}{\partial x}\nabla^2\psi_2+\frac{\lambda^2}{2}\left(U_2\frac{\partial\psi_1}{\partial x}-U_1\frac{\partial\psi_2}{\partial x}\right)\qquad(3.4)$$

$$+J(\psi_2,q_2)=-U_2\frac{\partial h}{\partial x},$$

where

$$q_1=\nabla^2\psi_1+\tfrac{1}{2}\lambda^2(\psi_2-\psi_1),\quad q_2=\nabla^2\psi_2+\tfrac{1}{2}\lambda^2(\psi_1-\psi_2)+h,$$

and U_1 and U_2 are the translating velocities of the upper and lower levels (1 and 2) respectively. λ is an inverse deformation radius, given by $\lambda=2f_0^2/\sigma\Delta p$, where σ is the static stability parameter $\rho^{-1}d(\ln\Theta_0)/dp$. Alternatively, in height co-ordinates $\lambda=f/Nh$, where N is the Brunt-Vaisala frequency.

Exact fully nonlinear solutions for (3.4) may be written down. They are

$$\psi_{1k}=\frac{\tfrac{1}{2}\lambda^2h_kU_1U_2}{[\beta^2-\beta(\lambda^2+2k^2)\bar{U}+\tfrac{1}{2}\lambda^2k^2(U_1^2+U_2^2)+k^4U_1U_2]},$$

$$\psi_{2k}=-2\psi_{1k}(\beta-\tfrac{1}{2}\lambda^2U_2-k^2U_1)/\lambda^2U_1\qquad(3.5)$$

where $\bar{U}=\tfrac{1}{2}(U_1+U_2)$. These are exact solutions because in both layers a functional relationship exists between q_i and ψ_i everywhere, namely $q_i=F_i\psi_i$ where

$$F_1=-[\beta-\tfrac{1}{2}\lambda^2(U_2-U_1)]/U_1,\quad F_2=-[\beta-\tfrac{1}{2}\lambda^2(U_1-U_2)]/U_2.\qquad(3.6)$$

The solutions are again the *only* solutions if, upstream or downstream far from the topography, the flow is purely zonal with velocities U_1 and U_2 in the upper and lower layers. Some interesting limits to (3.5) exist.

i) $k^2 \to 0$. This gives

$$\psi_1 = \tfrac{1}{2}\lambda^2 h U_2 U_1/(\beta^2 - \beta\lambda^2 \bar{U}),$$

with a similar expression for ψ_2. The condition for resonance depends now only on the mean barotropic flow structure. It occurs when $\frac{1}{2}(U_1 + U_2) = \beta/\lambda^2$. This problem may have considerable oceanic relevance since the oceanic deformation radius is so small.

ii) Barotropic Mean Field ($U_1 = U_2 = U$). The potential vorticity functionals simplify to $F_1 = F_2 = -\beta/U$, and the solutions for the stream function simplify to

$$\psi_{1\mathbf{k}} = \tfrac{1}{2} h_{\mathbf{k}} \lambda^2 U^2/(\beta - Uk^2)[\beta - U(k^2 + \lambda^2)]. \tag{3.7}$$

Two resonances now exist, at $\beta = Uk^2$ (a barotropic resonance) and $\beta = U(k^2 + \lambda^2)$ (a baroclinic resonance). Again both are the conditions for stationarity of the Rossby waves allowed in the system. At the barotropic resonance $\psi_1 = \psi_2$, whereas $\psi_1 = -\psi_2$ at the baroclinic resonance.

iii) Highly baroclinic mean field, i.e. $U_1 \gg U_2 \approx 0$. Since only the lower fluid level is directly affected by the topography, there is no topographic response (i.e. $\psi_1 = \psi_2 = 0$) unless the denominators in (3.5) are zero. This occurs when $\beta = U_1 k^2$ and taking this limit first (and then setting $U_2 = 0$) yields

$$\psi_{1\mathbf{k}} = h_{\mathbf{k}} U_1/\beta = -h_{\mathbf{k}}/k^2, \quad \psi_{2\mathbf{k}} = U_2 \psi_{1\mathbf{k}}/U_1.$$

Thus, for very small lower level flow, only the upper level is excited and only at the resonant wavenumber. The upper level response is finite and depends only on the topographic height and the wavenumber.

4. FLOW INSTABILITY—A LONG-WAVE APPROXIMATION

This section is the first of two which deal with the instability of flow over topography. We shall consider the stability of exact solutions of the equations describing barotropic and baroclinic (two-layer) flow over topography. The procedure is to specify a mean field U, then calculate the topographic response to this field ($\tilde{\psi}$, say) and then calculate the instability of this solution ($\tilde{\psi} - Uy$). This procedure is similar to, but not quite the same as, calculating the stability properties of a stationary state of a coupled viscous system such as Hart's or the Eqs. (2.3) and (2.4). In those systems viscosity and forcing are important in determining the stationary values of the zonal wind. Given this zonal wind and the resulting amplitude of the asymmetric components, one does not expect these forcings to play a major role in the development of any subsequent instabilities. In my analysis the zonal mean field is specified (although a zonal flow rectification is allowed, and in some cases is essential), and only the inviscid problem is considered. Nevertheless, unless viscosity is playing a very subtle role the essential instability mechanisms are the same in this case and in the forced, viscous case.

The stability equation is easily derived as follows. Suppose that an exact solution to the equations exists and is of the form $\tilde{Q} = \gamma\tilde{\Psi}$, where γ is some constant. Then the perturbation equation becomes, writing $\Psi = \tilde{\Psi} + \psi'$ and ignoring squares of primed quantities

$$\partial q'/\partial t + J(\tilde{\Psi}, q') + J(\psi', \gamma\tilde{\Psi}) = 0,$$

or

$$\partial q'/\partial t + (J\tilde{\Psi}, q' - \gamma\psi') = 0. \tag{4.1}$$

For a single Rossby wave of wavenumber $\mathbf{k}, \gamma = -k^2$; for barotropic topographic flow $\gamma = -\beta/U$.

The remainder of this section will invoke a long-wave approximation to enable explicit stability calculations of (4.1) to be performed. Unlike the conventional long-wave approximation, we shall consider motion only of large meridional scale. It is therefore appropriate for flow over topography with large meridional scale, such as long North–South ridge. [These are also the conditions which Hart (1979) considered.] However, the analysis below is presented as an explicit stability calculation, in which I perburb the exact solution to flow over topography, and examine only perturbations of large meridional scale.

4a. Barotropic flow

Let the basic flow be $\tilde{\Psi} = -Uy + \tilde{\psi}(x)$, where U is a constant zonal flow. For a Rossby wave then $\tilde{\Psi} = -Uy + Ae^{ikx}$ (plus complex conjugate) where U is a constant zonal flow and A is the arbitrary wave amplitude and k is the wavenumber. Note that the system is Galilean invariant, so we can choose $U = \beta/k^2$ so the Rossby wave is stationary. For flow over topography, $\tilde{\Psi} = -Uy + \tilde{\psi}(x)$ where $\tilde{\psi}(x)$ satisfies $\nabla^2\tilde{\psi} + \tilde{\psi}\beta/U = -h(x)$. In both cases (4.1) becomes, dropping the primes,

$$\partial(\nabla^2\psi)/\partial t + U(\partial/\partial x)(\nabla^2 + k_\beta^2)\psi + J(\tilde{\psi}, \nabla^2\psi + k_\beta^2\psi) = 0, \tag{4.2}$$

where $k_\beta^2 = \beta/U$. Now expand the perturbation stream function in powers of l, the meridional wavenumber nondimensionalized by some typical zonal horizontal scale k_0. Thus we seek instabilities of the form

$$\psi = [\psi_0(t) + (l/k_0)\psi_1(x,t) + (l/k_0)^2\psi_2(x,t) + \cdots]e^{ily}. \tag{4.3}$$

The coefficients are complex. Implicit always is the addition of complex conjugates. The reason for the scaling, and in particular for the zonal flow correction ψ_0 will become apparent below. l/k_0 is assumed much less than unity. Substituting (4.3) into (4.2) and equating powers of l yields, at order l and l^2 respectively:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1}{\partial x^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + k_\beta^2\right)\psi_1 + ik_\beta^2\tilde{\psi}_x\psi_0 k_0 = 0, \tag{4.4a}$$

$$k_0^2\frac{\partial\psi_0}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2\psi_2}{\partial x^2}\right) - U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + k_\beta^2\right)\psi_2 - ik_0\tilde{\psi}_x\left(k_\beta^2\psi_1 + \frac{\partial^2\psi_1}{\partial x^2}\right) = 0. \tag{4.4b}$$

If an explicit zonal flow correction is not allowed for in (4.3), (4.4a) would yield a stable Rossby wave. However, the last two terms on the left-hand side of (4.4b) in general do have a zonally averaged component. Without the term in ψ_0, there is nothing to balance them and there is an absurdity. Zonally averaging (4.4b) yields

$$k_0\,\partial\psi_0/\partial t - ik_\beta^2\overline{(\tilde{\psi}_x\psi_1)}^x - \overline{i\tilde{\psi}_x\,\partial^2\psi_1/\partial x^2}^x = 0. \tag{4.5}$$

This, and (4.4a) are a pair of coupled integro-differential equations whose eigenvalues determine the stability of (4.2). Combining the two equations yields

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2\psi_1}{\partial x^2}\right\} + U\frac{\partial}{\partial t}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + k_\beta^2\right)\psi_1 - k_\beta^2\tilde{\psi}_x\overline{\left[\tilde{\psi}_x\left(k_\beta^2 + \frac{\partial^2}{\partial x^2}\right)\psi_1\right]}^x. \tag{4.6}$$

Normal mode solutions are not generally valid. However, we may express $\tilde{\psi}$ and ψ_1 as Fourier series thus:

$$(\tilde{\psi}, \psi_1) = \sum_k (\tilde{\psi}_k, \psi_k)\,e^{ikx}.$$

Equation (4.6) becomes

$$-k^2\partial^2\psi_k/\partial t^2 + U(\partial/\partial t)(-ik^3 + ikk_\beta^2)\psi_k + ik\tilde{\psi}_k k_\beta^2 \sum_p ip\tilde{\psi}_{-p}\psi_p(k_\beta^2 - p^2) = 0. \tag{4.7}$$

There are two situations in which this may be solved analytically.

i) High wave amplitude limit Neglect the second term on the left-hand side of (4.7). Then seeking solutions of the form $\psi_k(t) = \psi_k e^{i\sigma t}$ for all k gives

$$k\sigma^2\psi_k - k_\beta^2\tilde{\psi}_k \sum_p p\tilde{\psi}_{-p}(k_\beta^2 - p^2)\psi_p = 0. \tag{4.8}$$

By inspection, we can write down one set of eigenfunctions and eigenvalues of this. They are

$$\psi_k \propto \tilde{\psi}_k/k, \quad \sigma^2 = -k_\beta^2 \sum_k |\tilde{\psi}_k|^2(k^2 - k_\beta^2). \tag{4.9}$$

From (4.9) and (4.7) the strong interaction or high wave amplitude limit can be seen to be consistent when $|p\tilde{\psi}_p| \gg |U|$. From (4.9), instability arises only when $k_\beta^2 > 0$. This is consistent with the Arnol'd criterion: when $\partial\tilde{q}/\partial\tilde{\psi} > 0$, a sufficient condition for stability is satisfied (See Section 5). If only one component of flow is present say of wavenumber m, instability arises when $(m^2 - \beta/U) > 0$, i.e. the flow on the "high side" of resonance is unstable. Equation (4.9) is a generalization of this result to more general topographic shapes.

ii) Single Fourier mode topography Equation (4.7) then becomes

$$-k^2\partial^2\psi_k/\partial t^2 + ikU(\partial/\partial t)(k_\beta^2 - k^2)\psi_k - k_\beta^2|k\tilde{\psi}_k|^2(k_\beta^2 - k^2)\psi_k = 0. \quad (4.10)$$

The dispersion relationship of (4.10) is

$$k^2\sigma^2 - kU\sigma q^2 - k_\beta^2 q^2 |k\tilde{\psi}_k|^2 = 0,$$

where $q^2 = k_\beta^2 - k^2$. This gives instability when all of the following conditions pertain:

$$k_\beta^2 > 0, \quad k_\beta^2 < k^2, \quad |k\tilde{\psi}_k|^2 > -U^2q^2/4k_\beta^2.$$

For a Rossby wave, $k_\beta = k$ and no instability exists. Instability does exist for eastward superresonant flow over topography, provided the topographic wave is of large enough amplitude. This will be referred to as "form-drag instability". It may be considered a triad-instability, in which the three interacting components are the zonal flow, the topography and a component close to the topographic scale. Other triads are considered in Section 5.

4b. Two-layer flow

The equations governing the growth of perturbations about the steady solution (3.5) are readily written down. The two-layer version of (4.1) is (dropping the primes)

$$\frac{\partial q_1}{\partial t} + \beta\frac{\partial\psi_1}{\partial x} + U_1\frac{\partial}{\partial x}\nabla^2\psi_1 + \frac{\lambda^2}{2}\left\{U_1\frac{\partial\psi_2}{\partial x} - U_2\frac{\partial\psi_1}{\partial x}\right\}$$

$$+ J(\tilde{\psi}_1, q_1 - F_1\psi_1) = 0,$$

$$\frac{\partial q_2}{\partial t}+\beta\frac{\partial \psi_2}{\partial x}+U_2\frac{\partial}{\partial x}\nabla^2\psi_2+\frac{\lambda^2}{2}\left\{U_2\frac{\partial \psi_1}{\partial x}-U_1\frac{\partial \psi_1}{\partial x}\right\}$$

$$+J(\tilde{\psi}_2, q_2-F_2\psi_2)=0, \tag{4.11}$$

with F_1 and F_2 given by (3.6) and q_1 and q_2 given by (3.4b). Writing $\psi=\frac{1}{2}(\psi_1+\psi_2)$, $\tau=\frac{1}{2}(\psi_1-\psi_2)$ we find

$$\frac{\partial}{\partial t}\nabla^2\psi+\tfrac{1}{2}(U_1+U_2)\frac{\partial}{\partial x}(k_\psi^2+\nabla^2)\psi+\tfrac{1}{2}(U_1-U_2)\frac{\partial}{\partial x}\nabla^2\tau$$

$$+J(\tilde{\psi}, q_1-F_1\psi_1+q_2-F_2\psi_2)+J(\tilde{\tau}, q_1-F_1\psi_1-q_2+F_2\psi_2)=0, \tag{4.12a}$$

$$\frac{\partial}{\partial t}(\nabla^2-\lambda^2)\tau+\tfrac{1}{2}(U_1-U_2)\frac{\partial}{\partial x}(\nabla^2\psi+\lambda^2\psi)+\tfrac{1}{2}(U_1+U_2)\frac{\partial}{\partial x}(\nabla^2\tau+k_\tau^2\tau)$$

$$+J(\tilde{\psi}, q_1-F_1\psi_1-q_2+F_2\psi_2)+J(\tilde{\tau}, q_1-F_1\psi_1+q_2-F_2\psi_2)=0, \tag{4.12b}$$

where $k_\psi^2=k_\beta^2, k_\tau^2=k_\beta^2-\lambda^2$.

Seeking the form-drag instability, we may attempt a perturbation expansion along the lines of (4.3). This turns out to be valid for ψ. For τ we would have then

$$\tau=[\tau_0(t)+(l/k_0)\tau_1(x,t)+(l/k_0)^2\tau_2(x,t)+\cdots]e^{ily}.$$

At zero order we obtain the trivial result

$$d(-2\lambda^2\tau_0)/dt=0.$$

Therefore, directly from the scaling, it is clear that form-drag instability acts only to produce a barotropic correction to the zonally averaged field. [A related, but less general, result was obtained by Charney and Flierl (1981) by a somewhat different method and with a purely zonal basic state.] It would be possible to obtain from (4.11) a dispersion relationship for arbitrary shear, and solve this numerically. However, it is more instructive to specialize to the case $U_1 = U_2 = U$ (which is manifestly stable in the absence of topography), for then the instability arises in its purest form, and show that topography may destabilize the flow both baroclinically and barotropically. Equations (4.12) become

$$\partial(\nabla^2\psi)/\partial t + U(\partial/\partial x)(k_\psi^2 + \nabla^2)\psi + J(\tilde{\psi}, \nabla^2\psi + k_\psi^2\psi) + J(\tilde{\tau}, \nabla^2\tau + k_\tau^2\tau) = 0,$$

$$(\partial/\partial t)(\nabla^2 - \lambda^2)\tau + U(\partial/\partial x)(k_\tau^2 + \nabla^2)\tau$$

$$+ J[\tilde{\psi}, (\nabla^2 + k_\tau^2)\tau] + J(\tilde{\tau}, \nabla^2\psi + k_\psi^2) = 0.$$

Effecting the following expansion in powers of $l'(=l/k_0)$

$$\psi = [\psi_0(t) + l'\psi_1(x,t) + \dots]e^{ily}, \quad \tau = [l'\tau_1(x,t) + \dots]e^{ily},$$

yields

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \lambda^2\right)\tau_1 + U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + k_\tau^2\right)\tau_1 + i\tilde{\tau}_x k_\psi^2 \psi_0 k_0 = 0, \tag{4.13a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi_1}{\partial x^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + k_\psi^2\right)\psi_1 + i\tilde{\psi}_x k_\psi^2 \psi_0 k_0 = 0, \tag{4.13b}$$

$$-k_0\frac{\partial}{\partial t}\psi_0 + i\tilde{\psi}_x\overline{\left(\frac{\partial^2}{\partial x^2} + k_\psi^2\right)\psi_1}^x + i\tilde{\tau}_x\overline{\left(\frac{\partial^2}{\partial x^2} + k_\tau^2\right)\tau_1}^x = 0. \tag{4.13c}$$

Considering the problem as one of interacting "triads", only two possible types of interactions are possible, namely $(\tilde{\psi}, \psi_1, \psi_0)$ and $(\tilde{\tau}, \tau_1, \psi_0)$. The first is a purely barotropic triad, the second a mixed-mode instability involving two baroclinic modes and the barotropic zonal flow. The baroclinic modes evidently interact in the same way as the barotropic modes, with the familiar replacement in wave-number $k^2 \to k^2 + \lambda^2$ (e.g. Vallis, 1983). For the case $|\tilde{\psi}| \gg |\tilde{\tau}|$, (4.13b and c) reduce to the barotropic system (4.4). This condition holds near the barotropic resonance $U = \beta/k^2$. The instability is purely barotropic. On the other hand, near the baroclinic resonance $U = \beta/(k^2 + \lambda^2)$, $|\tilde{\tau}| \gg |\tilde{\psi}|$. Equations (4.13) then reduce to

$$(\partial/\partial t)(\partial^2/\partial x^2-\lambda^2)\tau_1+U(\partial/\partial x)(\partial^2/\partial x^2+k_\tau^2)\tau_1+i\tilde{\tau}_x k_\psi^2\psi_0 k_0=0,$$

$$k_0\partial\psi_0/\partial t-\overline{i\tilde{\tau}_x(\partial^2/\partial_x^2+k_\tau^2)\tau_1}^{x}=0. \tag{4.14}$$

This is identical with (4.4), except that the wavenumber k^2 is replaced by the pseudo-wavenumber $k'^2=k^2+\lambda^2$ wherever it appears via the ∇^2 operator on τ. The analysis now proceeds precisely as in Section 4a, now with potential baroclinic instabilities arising. Eliminating ψ_0 from (4.14) gives

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2}-\lambda^2\right)\tau_1+\frac{\partial}{\partial t}U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+k_\tau^2\right)\tau_1-k_\psi^2\tilde{\tau}_x\overline{\tilde{\tau}_x\left(\frac{\partial^2}{\partial x^2}-\lambda^2+k_\beta^2\right)\tau_1}^{x}=0. \tag{4.15}$$

In the high wave amplitude limit (4.15) gives, after Fourier transforming:

$$\sigma^2(k^2+\lambda^2)\tau_k-kk_\psi^2\tilde{\tau}_k\sum_p p\tilde{\tau}_{-p}[k_\beta^2-(p^2+\lambda^2)]\tau_p=0,$$

with eigenfunctions

$$\tau_k\propto k\tilde{\tau}_k/(k^2+\lambda^2)$$

and eigenvalues

$$\sigma^2=k_\beta^2\sum_p p^2\tilde{\tau}_p^2[k_\beta^2-(p^2+\lambda^2)]/(p^2+\lambda^2). \tag{4.16}$$

For a single Fourier mode topography we obtain

$$(k^2+\lambda^2)\sigma^2-\sigma Uk[k_\beta^2-(k^2+\lambda^2)]-k_\beta^2|k\tilde{\tau}_k|^2[k_\beta^2-(k^2+\lambda^2)]=0. \tag{4.17}$$

As in the barotropic case instability exists only for eastward super-resonant flow (now defined by $k^2+\lambda^2>\beta/U$), or its appropriate generalisation (4.16) if the topography has many Fourier modes. The energy cycle for such an instability is zonal barotropic instability + stationary eddy baroclinic energy→transient (growing) eddy baroclinic and zonal barotropic energy. In the case with vertical shear the topographic instability is combined with a conventional instability drawing energy from the zonal shear.

5. GENERAL STABILITY CONSIDERATIONS

5a. Integral constraints and nonlinear stability

This subsection examines some of the effects which Arnold-type constraints—imposed essentially by the conservation of energy and enstrophy—have on the stability problem. The Arnold stability criterion (Arnold, 1965; Blumen, 1968; Pierini and Vulpiani, 1981; Holm *et al.*, 1985) is a sufficient criterion for nonlinear stability. The general method involves taking variations of some conserved functional of the streamfunction, and under certain conditions nonlinear stability can be demonstrated. Because of the simple, linear relationships here between q and ψ it is easier, and instructive, to present an equivalent argument directly from the equations of motion.

The stationary solutions to the quasi-geostrophic equations discussed earlier are all of the form $\tilde{Q}=\gamma(z)\tilde{\Psi}$ where z is a vertical co-ordinate. The nonlinear perturbation equation is then (with $Q=\tilde{Q}+q; \Psi=\tilde{\Psi}+\psi$):

$$\partial q/\partial t + J[\tilde{\Psi}, q-\gamma(z)\psi] + J(\psi, q) = 0. \tag{5.1}$$

Thus, q and ψ are the perturbation potential vorticity and streamfunction respectively.) Perturbation energy and enstrophy equations are formed by multiplying (5.1) by $-\psi$ and q. If $q=\nabla^2\psi+\lambda^2\partial^2\psi/\partial z^2$ (where λ is an appropriate inverse deformation radius) then we find

$$\tfrac{1}{2}(d/dt)\langle(\nabla\psi)^2+\lambda^2(\partial\psi/\partial z)^2\rangle - \langle\psi J(\tilde{\Psi}, q)\rangle = 0, \tag{5.2}$$

and

$$\tfrac{1}{2}(d/dt)\langle q^2/\gamma\rangle - \langle qJ(\tilde{\Psi}, \psi)\rangle = 0, \tag{5.3}$$

where $\langle\ \rangle$ denotes a volume integration. Equation (5.2) and (5.3) are valid in doubly periodic or channel domains, or in an infinite domain with the perturbation streamfunction vanishing at infinity. I have also assumed $\partial\psi/\partial z$ vanishes at the upper and lower boundaries. [See Blumen (1968) for the effects of nonzero $\partial\psi/\partial z$.] Adding (5.2) and (5.3) gives

$$(d/dt)\langle(\nabla\psi)^2+\lambda^2(\partial\psi/\partial z)^2+\gamma^{-1}q^2\rangle = 0. \tag{5.4}$$

If the integrand in (5.4) is positive or negative definite, the perturbation cannot grow and the flow is stable, in the sense of Liapunov. One sufficient condition for positive definiteness (Arnold's condition) is that γ be everywhere positive.

In layered-models, and finite difference versions of the continuous equations, the definition of q_i (the potential vorticity in the ith layer) *includes* the boundary conditions. Thus at an interior layer "i" we should have

$$q_i=\nabla^2\psi_i+\tfrac{1}{2}\lambda^2(\psi_{i+1}+\psi_{i-1}-2\psi_i), \tag{5.5}$$

whereas at the lowest level "N":

$$q_N=\nabla^2\psi_N+\tfrac{1}{2}\lambda^2(\psi_{N-1}-\psi_N)+h. \tag{5.6}$$

Given (5.5) and (5.6), the layered version of (5.4) is simply

$$(d/dt)\sum_{i=1}^{N}\langle(\nabla\psi_i)^2+\tfrac{1}{2}\lambda^2(\psi_i-\psi_{i+1})^2+\gamma_i^{-1}q_i^2\rangle=0, \tag{5.7}$$

provided I define $\psi_{N+1}=\psi_N$, and $\langle\ \rangle$ is now only a horizontal integration.

For a two-level model (5.7) becomes

$$(d/dt)\langle(\nabla\psi_1)^2+(\nabla\psi_2)^2+\tfrac{1}{2}\lambda^2(\psi_1-\psi_2)^2+\gamma_1^{-1}q_1^2+\gamma_2^{-1}q_2^2\rangle=0. \tag{5.8}$$

Before returning to topographic effects, I shall show that (5.4) and (5.7) give useful insights into ordinary baroclinic instability. For two layer flow with basic state

$$\tilde{\Psi}_1=U_1y,\qquad \tilde{Q}_1=\beta y-\tfrac{1}{2}\lambda^2(U_2-U_1)y,$$

$$\tilde{\Psi}_2=-U_2y,\quad \tilde{Q}_2=\beta y-\tfrac{1}{2}\lambda^2(U_1-U_2)y.$$

We evidently have

$$\gamma_1=-(\beta+\lambda^2U)/(U_0+U),\quad \gamma_2=-(\beta-\lambda^2U)/(U_0-U),$$

where

$$U_1=U_0+U,\quad U_2=U_0-U.$$

We can *choose* $U_0 < -U$ (since a Galilean translation does not alter stability properties). Then both γ_1 and γ_2 are positive if $\beta > \lambda^2 U$ and the flow is stable. The condition is just that which can be obtained, less generally, by a normal mode analysis. Therefore even if the flow is turbulent, energy and enstrophy cannot be extracted from the mean flow if the shear is sub-critical ($U < \beta/\lambda^2$). The condition that necessarily stable zonal flows be pseudo-westward everywhere (cf. Andrews, 1984) is not violated, because of our freedom to choose U_0.

If we let $U_0 = \beta/\lambda^2$, then $\gamma_1 = \gamma_2 = -\lambda^2$. The constraint (5.8) then takes the nicely symmetric form:

$$(d/dt)\sum_{\mathbf{k}} \{k^2 \psi_{\mathbf{k}}^2 (k^2 - \lambda^2) + k'^2 \tau_{\mathbf{k}}^2 (k'^2 - \lambda^2)\} = 0, \tag{5.9}$$

where $\psi = \frac{1}{2}(\psi_1 + \psi_2)$, $\tau = \frac{1}{2}(\psi_1 - \psi_2)$, $k'^2 = k^2 + \lambda^2$ and the stream function has been spectrally expanded over all k:

$$[\psi(x, y), \tau(x, y)] = \sum_{\mathbf{k}} (\psi_{\mathbf{k}}, \tau_{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{x}).$$

In the linear problem (which, because the basic flow is purely zonal, does not cause perturbation modes to interact), modes for which $k > \lambda$ are stable, whatever the shear, since if the sum in (5.9) is restricted to $k > \lambda$ it is positive definite. If the smallest allowable wavenumber is greater than λ (say for geometrical reasons), the flow is always (nonlinearly) stable, whatever the shear. Physically, the mean flow cannot transfer both energy and enstrophy to a mode of wavenumber greater than λ and still satisfy the attendant constraints.

Similar constraints are useful in multi-layered systems also, especially if the mean flow has a simple vertical structure. For consider the problem with a mean vertical shear given by $\tilde{\Psi} = Uy \cos pz$ where the vertical coordinate is scaled so that the top of atmosphere is at $z = \pi$, and rigid lids with $\partial\psi/\partial z = 0$ exist here and at $z = 0$. It is readily shown that the flow is stable for $U < \beta/\lambda^2 p^2$. Hence the greater the vertical structure of the basic flow, the smaller the critical shear for instability. Appending to the shear flow a dynamically inactive barotropic flow of the form $\beta y/\lambda^2 p^2$ gives a constant $\gamma = \tilde{q}/\tilde{\psi} = -\lambda^2 p^2$ everywhere. Expand the perturbation streamfunction as

$$\psi = \sum_{\mathbf{k},\alpha} \psi_{\mathbf{k}\alpha} \exp(i\mathbf{k} \cdot \mathbf{x}) \cos \alpha z. \tag{5.10}$$

The constraint (5.4) becomes

$$(d/dt)\sum_{\mathbf{k},\alpha}\psi_{\mathbf{k}\alpha}^2\{(k^2+\lambda^2\alpha^2)[\lambda^2p^2-(k^2+\lambda^2\alpha^2)]\}=0.$$

If the basic flow is simple (say $p=1$) then stability is assured if $(k^2+\lambda^2\alpha^2)>\lambda^2$ for all k,α. This is guaranteed if $k^2>\lambda^2$, so again, for this simple problem, there is a high wavenumber cut-off to instability. Furthermore, note that the growing structure must have a barotropic component ($\alpha=0$) which feels the presence of both upper and lower boundaries. If $p>1$, a greater range of wavenumbers is unstable. Physically, there is more enstrophy in the basic state and higher wavenumbers can be unstable. These conclusions are of course valid only for disturbances which satisfy the isothermal boundary conditions. A uniform shear ($\Psi=-Uyz$) is always non-linearly stable with such boundary conditions.

5b. Integral constraints on topographic instability

The minimum shear required for instability (β/λ^2) in the two layer case does not apply if topography is present, because the flow is no longer Galilean invariant and U_0 is no longer arbitrary. However, the constraints do impose some necessary conditions for instability. Consider first barotropic flow over topography, with $\tilde{q}=-\beta\tilde{\psi}/U$. The integral constraint (5.4) reduces to

$$(d/dt)\langle(\nabla\psi)^2-k_\beta^{-2}(\nabla^2\psi)^2\rangle=0. \tag{5.11}$$

Easterly flow ($U<0$) is therefore necessarily stable. Expanding the streamfunction spectrally enables (5.11) to be written

$$(d/dt)\sum\psi_{\mathbf{k}}^2k^2(k_\beta-k^2)=0. \tag{5.12}$$

Form-drag instability may be considered a triad-interaction involving the zonal flow ($k<k_\beta$), the stationary topographic wave and another mode of similar scale to it, say wavenumber k'. Thus (5.12) is positive definite (and the flow stable) if $k_\beta^2>k'^2$, or $U<\beta/k'^2$. Flow on the "low-side" of resonance is thus stable. The stability arises entirely because the meridional scale of the topography is assumed large (i.e. $l\ll k$). If l is not negligible flow on the low-side is not necessarily stable (as found by explicit calculation below). The general condition which emerges from (5.12) is this: A stationary flow over topography, or a stationary Rossby wave, may be unstable to a triad interaction provided one triad member (m) is such that $k_\beta > m$ and the other (n) is such that $k_\beta < n$. In two layer flow with no shear (so $\gamma_1 = \gamma_2 = -\beta/U$) the analogous condition to (5.12) is

$$(d/dt)\sum\{\psi_{\mathbf{k}}^2 k^2(k^2 - k_\beta^2) + \tau_{\mathbf{k}}^2 k'^2(k'^2 - k_\beta^2)\} = 0.$$

Consider triad-interactions involving a barotropic basic state, $\tilde{\psi}_{\mathbf{k}}$, or a baroclinic basic state $\tilde{\tau}_{\mathbf{k}}$. The allowable triads are, in the former case, $(\tilde{\psi}_{\mathbf{k}}, \psi_{\mathbf{p}}, \psi_{\mathbf{q}})$, $(\tilde{\psi}_{\mathbf{k}}, \tau_{\mathbf{p}}, \tau_{\mathbf{q}})$ and in the latter case $(\tilde{\tau}_{\mathbf{k}}, \tau_{\mathbf{p}}, \psi_{\mathbf{q}})$ where in all cases $\mathbf{p} + \mathbf{q} = \mathbf{k}$. Stability is guaranteed unless $p^2 > k_\beta^2$ and $q^2 < k_\beta^2$ or $p^2 < k_\beta^2$ and $q^2 > k_\beta^2$, with the replacement $p^2, q^2 \rightarrow p^2 + \lambda^2, q^2 + \lambda^2$ if the mode involved is baroclinic. The point is that triads involving a baroclinic member behave very similarly to purely barotropic triads, provided the rescaling $k^2 \rightarrow k^2 + \lambda^2$ on baroclinic components is performed, and that an instability must involve modes with wavenumbers, or "pseudo-wavenumbers", on either side of k_β.

5c. Explicit stability calculations

This section considers the stability properties of one and two layer flow over topography when the meridional wavenumber is not small. To abstract the physical mechanisms a number of simplifications are made. First, the topography will be allowed to have only one wavenumber (although this is *not* a condition for a stationary solution of the equations). Second, only triad interactions involving the basic state and two interacting modes are initially considered. Third, in the two layer case no vertical shear in the zonally averaged state is allowed which, as in Section 4, allows the instability to be displayed in its purest form.

i) Barotropic flow This is considered as a preliminary to the two-layer case. Perturbations to a zonally asymmetric basic state $\tilde{\psi}$, set up by a uniform zonal current U flowing over topography $h(x, y)$, satisfy (4.2), namely

$$\partial(\nabla^2\psi)/\partial t + U(\partial/\partial x)(\nabla^2 + k_\beta^2)\psi + J[\tilde{\psi}, (\nabla^2 + k_\beta^2)\psi] = 0, \quad (5.13)$$

where $\tilde{\psi}$ satisfies

$$\nabla^2\tilde{\psi}+k_\beta^2\tilde{\psi}=-h(x,y).$$

Equation (5.13) is precisely that studied by Gill (1974) in his study of the stability of Rossby waves, provided one allows the basic state $\tilde{\psi}$ to comprise only wavenumber $\mathbf{k}$ where $|\mathbf{k}|=k_\beta$. Gill's approach was to expand the perturbation streamfunction as:

$$\psi=\psi_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}+\psi_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}+\psi_{\mathbf{r}}e^{i\mathbf{r}\cdot\mathbf{x}}+\cdots,$$

where $\mathbf{k}+\mathbf{p}=\mathbf{r}$ and $\mathbf{k}-\mathbf{p}=\mathbf{q}$. The truncation is applied after the three terms shown, so two triads are involved, namely $(\mathbf{k},\mathbf{p},\mathbf{q})$ and $(\mathbf{k},\mathbf{p},\mathbf{r})$. The two parameters of the problem are the primary wave $(\tilde{\psi}_{\mathbf{k}})$ amplitude and its direction. The truncation is to some extent arbitrary: in the weak interaction or low wave amplitude limit $(k_\beta^3\tilde{\psi}\ll\beta)$ the truncation is valid. In the high wave amplitude limit it is not, although the growth rates (of mode p) are the right order of magnitude. To display the triadic nature of the instability it is sufficient to consider the simplest problem which is that of just one interacting triad, say $\tilde{\psi}_{\mathbf{k}},\psi_{\mathbf{p}},\psi_{\mathbf{q}}$. (In the Rossby-wave case, only one wave is *allowed* in the basic state for it still to be a solution of the equations of motion. Although this is not the case in the topographic problem it is a useful simplification.) Assuming then that $\tilde{\psi}=\tilde{\psi}_{\mathbf{k}}e^{\mathbf{k}\cdot\mathbf{x}}$ and $\psi=\psi_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}+\psi_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}$ where $-\mathbf{k}+\mathbf{p}=\mathbf{q}$ the equations governing the evolution of $\psi_{\mathbf{p}}$ and $\psi_{\mathbf{q}}$, obtained from (5.13) are:

$$d(-p^2\psi_{\mathbf{p}})/dt+i\omega_{\mathbf{p}}p^2\psi_{\mathbf{p}}+a_{pkq}\tilde{\psi}_{\mathbf{k}}(k_\beta^2-q^2)\psi_{\mathbf{q}}=0,$$

$$d(-q^2\psi_{\mathbf{q}})/dt+i\omega_{\mathbf{q}}q^2\psi_{\mathbf{q}}+a_{q-kp}\tilde{\psi}_{-\mathbf{k}}(k_\beta^2-p^2)\psi_{\mathbf{p}}=0, \tag{5.14}$$

where $\omega_p=p_x(\beta-Up^2)$ and similarly for ω_q, and $a_{pkq}=-(\mathbf{k}\times\mathbf{q})$. Equation (5.14) yields the dispersion relationship (letting $\psi_{\mathbf{p}}=\psi_{o\mathbf{p}}e^{i\sigma t}$ etc.):

$$\sigma^2-\sigma(\omega_p+\omega_q)+\omega_p\omega_q+a_{pkq}a_{q-kp}\tilde{\psi}_k^2(k_\beta^2-q^2)(k_\beta^2-p^2)/p^2q^2=0. \tag{5.15}$$

Now, $a_{pkq}=(\mathbf{k}\times\mathbf{q})$ and $a_{qkp}=-(\mathbf{k}\times\mathbf{p})=-(\mathbf{k}\times\mathbf{q})$. Instability (negative σ^2) therefore only arises if one (and only one) of p^2 or q^2 is larger than k_β^2, and the other is smaller. Furthermore, k_β^2 must be positive. Before discussing the case further, we turn to the two-layer situation.

ii) Two layer baroclinic instability over topography. The related problem of the stability of baroclinic Rossby waves was considered by Jones (1978). In general the mean flow now consists of, as well as the zonally averaged flow $-Uy$ in each layer, a baroclinic, $\tilde{\tau}$, and barotropic, $\tilde{\psi}$, zonally asymmetric state. The perturbation equations are derived from (4.12) and are:

$$\partial(\nabla^2\psi)/\partial t+U(\partial/\partial x)(k_\beta^2+\nabla^2)\psi+J(\tilde{\psi},\nabla^2\psi+k_\beta^2\psi)$$

$$+J[\tilde{\tau},(\nabla^2-\lambda^2)\tau+k_\beta^2\tau]=0,$$

$$(\partial/\partial t)(\nabla^2-\lambda^2)\tau+U(\partial/\partial x)[k_\beta^2+(\nabla^2-\lambda^2)\tau]+J[\tilde{\psi},(\nabla^2-\lambda^2)\tau+k_\beta^2\tau]$$

$$+J(\tilde{\tau},\nabla^2\psi+k_\beta^2\psi)=0. \qquad (5.16)$$

There is a great deal of similarity between (5.16) and (5.14). Allowing the stationary flow to exist at only one wavenumber, $\mathbf{k}$, and perturbations at wavenumber $\mathbf{p}$ and $\mathbf{q}$ the interacting triads are: $(\psi_\mathbf{k},\psi_\mathbf{p},\psi_\mathbf{q})$, the barotropic triad; and $(\tilde{\psi}_\mathbf{k},\tau_\mathbf{p},\tau_\mathbf{q})$ and $(\tilde{\tau}_\mathbf{k},\psi_\mathbf{p},\tau_\mathbf{q})$ which are mixed triads. (Form-drag instability, like conventional baroclinic instability, may be considered a non-local triad interaction around an isosceles triangle with $|\mathbf{k}|\approx|\mathbf{q}|\gg|\mathbf{p}|$. However, in that case the interaction $(\tilde{\psi}_\mathbf{k},\tau_\mathbf{p},\tau_\mathbf{q})$ does not exist because the zonal flow correction is barotropic).

The relative magnitudes of $\tilde{\psi}_\mathbf{k}$ and $\tilde{\tau}_\mathbf{k}$ are determined by the value of U. From (3.5):

$$\tilde{\psi}_\mathbf{k}=h_\mathbf{k}/(k_\beta^2-k^2), \quad \tilde{\tau}_\mathbf{k}=h_\mathbf{k}/[k_\beta^2-(k^2+\lambda^2)]. \qquad (5.17)$$

At the baroclinic resonance, $U=\beta/(\lambda^2+k^2)$, the barotropic response remains finite, and conversely at the barotropic resonance. For a topography consisting of a given wavenumber component, then, the relative magnitudes of the baroclinic and barotropic stationary flow components are determined by the relative magnitudes of $k_\beta^2-(\lambda^2+k^2)$ and $k_\beta^2-k^2$ respectively. The stability properties of $\tilde{\tau}_{\mathbf{k}}$ and $\tilde{\psi}_{\mathbf{k}}$ are distinct, and may be considered separately. The stability of $\tilde{\psi}_{\mathbf{k}}$ to purely barotropic perturbations is governed by (5.16a), with $\tilde{\tau}_{\mathbf{k}}=0$, which is then identical to (5.13). The stability of barotropic and mixed interactions [i.e. $(\tilde{\psi}_{\mathbf{k}}, \tau_{\mathbf{p}}, \tau_{\mathbf{q}})$ and $(\tilde{\tau}_{\mathbf{k}}, \psi_{\mathbf{p}}, \tau_{\mathbf{q}})$] is governed by, in all cases:

$$(d/dt)(-p'^2\phi_{\mathbf{p}})+i\omega_{\mathbf{p}}p'^2\phi_{\mathbf{p}}+a_{pkq}\tilde{\Phi}_{\mathbf{k}}(k_\beta^2-q'^2)\phi_{\mathbf{q}}=0, \qquad (5.18a)$$

$$(d/dt)(-q'^2\phi_{\mathbf{q}})+i\omega_{\mathbf{q}}q'^2\phi_{\mathbf{q}}+a_{q-kp}\tilde{\Phi}_{-\mathbf{k}}(k_\beta^2-q'^2)\phi_{\mathbf{p}}=0. \qquad (5.18b)$$

Here, $\tilde{\Phi}_{\mathbf{k}}=\tilde{\tau}_{\mathbf{k}}$ or $\tilde{\psi}_{\mathbf{k}}$, $q'^2=q^2$ or $q^2+\lambda^2$, $\phi_{\mathbf{p}}=\psi_{\mathbf{p}}$ or $\tau_{\mathbf{p}}$, $\omega_{\mathbf{p}}=Up_x(k_\beta^2-p^2)/p^2$ or $Up_x[k_\beta^2-(p^2+\lambda^2)]/(p^2+\lambda^2)$ and $\omega_{\mathbf{q}}=Uq_x(k_\beta^2-q^2)/q^2$ or $Uq_x[k_\beta^2-(q^2+\lambda^2)]/(q^2+\lambda^2)$. The allowable combinations are listed in Table I. Equations (5.18) and (5.14) are identical, and so the dispersion relationship resulting from (5.18) is the same as (5.15). The next subsection discusses the barotropic and baroclinic stability properties together.

5d. Barotropic and baroclinic instability—discussion

The generalized dispersion relationship from (5.18) or (5.14) is:

$$\sigma^2-\sigma(\omega_p+\omega_q)+\omega_p\omega_q+a_{pkq}a_{qkp}\tilde{\Phi}_{\mathbf{k}}^2(k_\beta^2-q'^2)(k_\beta^2-p'^2)/p'^2q'^2=0. \qquad (5.19)$$

The following seem to be salient points:

i) The problem of the stability of a Rossby wave is the same as the problem of the stability of the stationary flow resulting from a uniform zonal current flowing over topography of a single wavenumber $(\beta/U)^{1/2}$.

ii) The one and two layer problems may be put into formally identical forms.

iii) The instability needs only *two* other modes, aside from the topographic mode, in order to exist. That is to say, the nature of the

TABLE I

Various values of parameters in (5.18) for the four types of triad interaction: $(\tilde{\psi}_{\mathbf{k}}, \psi_{\mathbf{p}}, \psi_{\mathbf{q}})$, $(\tilde{\psi}_{\mathbf{k}}, \tau_{\mathbf{p}}, \tau_{\mathbf{q}})$, $(\tilde{\tau}_{\mathbf{k}}, \tau_{\mathbf{p}}, \psi_{\mathbf{q}})$, $(\tilde{\tau}_{\mathbf{k}}, \psi_{\mathbf{p}}, \tau_{\mathbf{q}})$, and $(\tilde{\tau}_{\mathbf{k}}, \tau_{\mathbf{p}}, \psi_{\mathbf{q}})$. In (5.20) mode **r** always has the same character (baroclinic or barotropic) as mode **q**. The comments in the first row of the table refer to the case when mode **p** is the "primary perturbation", and modes **q** and **r** the "secondary perturbations".

Parameters	Barotropic perturbations on a barotropic basic state	Baroclinic perturbations on a barotropic basic state	Baroclinic perturbations on a baroclinic basic state	Barotropic perturbations on a baroclinic basic state
$\tilde{\phi}_{\mathbf{k}}, \phi_{\mathbf{p}}, \phi_{\mathbf{q}}$	$\tilde{\psi}_{\mathbf{k}}, \psi_{\mathbf{p}}, \psi_{\mathbf{q}}$	$\tilde{\psi}_{\mathbf{k}}, \tau_{\mathbf{p}}, \tau_{\mathbf{q}}$	$\tilde{\tau}_{\mathbf{k}}, \tau_{\mathbf{p}}, \psi_{\mathbf{q}}$	$\tilde{\tau}_{\mathbf{k}}, \psi_{\mathbf{p}}, \tau_{\mathbf{q}}$
p'^2	p^2	$p^2+\lambda^2$	$p^2+\lambda^2$	p^2
q'^2	q^2	$q^2+\lambda^2$	q^2	$q^2+\lambda^2$
ω_p	$\dfrac{Up_x(k_\beta^2-p^2)}{p^2}$	$\dfrac{Up_x[k_\beta^2-(p^2+\lambda^2)]}{(p^2+\lambda^2)}$	$\dfrac{Up_x[k_\beta^2-(p^2+\lambda^2)]}{(p^2+\lambda^2)}$	$\dfrac{Up_x(k_\beta^2-p^2)}{p^2}$
ω_q	$\dfrac{Uq_x(k_\beta^2-q^2)}{q^2}$	$\dfrac{Uq_x[k_\beta^2-(q^2+\lambda^2)]}{(q^2+\lambda^2)}$	$\dfrac{Uq_x(k_\beta^2-q^2)}{q^2}$	$\dfrac{Uq_x[k_\beta^2-(q^2+\lambda^2)]}{(q^2+\lambda^2)}$

instability is a triad interaction between the stationary flow set up by the topography (and of the same wavenumber) and two "free" modes with wavenumbers (or pseudo-wavenumbers) on either side of k_β.

iv) No mean (zonally-averaged) vertical shear is needed for baroclinic instability. The instability is, or can be "baroclinic" because of the existence of growing modes with baroclinic vertical structures. Even if the basic state in purely barotropic ($\tilde{\tau}=0$), both eddy baroclinic and barotropic energies may grow. The former is unlikely, however, since this requires the existence of modes p and q such that $p^2+\lambda^2 \leqq k_\beta^2 \leqq q^2+\lambda^2$. Typically in the earth's atmosphere $k_\beta \approx 3$ or 4 (in nondimensional wavenumber units) whereas $\lambda \approx 8$ or 10.

All of the perturbation modes—the set $\{\psi_\mathbf{p}\}$—are coupled through the Jacobian. The general form of the instability is therefore a matrix eigenvalue problem with as many eigenvalues as there are modes in the system. Ultimately all of the modes grow at the same rate—that of the largest eigenvalue. Initially though, the growth rate of a particular mode $\psi_\mathbf{p}$ is determined through its interaction with $\psi_\mathbf{k}$ and those modes which directly interact with $\psi_\mathbf{k}$ and $\psi_\mathbf{p}$, namely those with wavevectors $\mathbf{q}=\mathbf{p}-\mathbf{k}$ and $\mathbf{r}=\mathbf{p}+\mathbf{k}$. The instability is then governed by [cf. (5.18)]:

$$(d/dt)(-p'^2\phi_\mathbf{p})+i\omega_\mathbf{p}p'^2\phi_\mathbf{p}+a_{pkq}\bar{\Phi}_\mathbf{k}(k_\beta^2-q'^2)\psi_\mathbf{q}+a_{pkr}\bar{\Phi}_{-\mathbf{k}}(k_\beta^2-r'^2)\phi_\mathbf{r}=0,$$

$$(d/dt)(-q'^2\phi_\mathbf{q})+i\omega_\mathbf{q}q'^2\phi_\mathbf{q}+a_{q-kp}\bar{\Phi}_{-\mathbf{k}}(k_\beta^2-p'^2)\phi_\mathbf{p}=0,$$

$$(d/dt)(-r'^2\phi_\mathbf{r})+i\omega_\mathbf{r}r'^2\phi_\mathbf{r}+a_{rkp}\bar{\Phi}_\mathbf{k}(k_\beta^2-p'^2)\phi_\mathbf{p}=0. \tag{5.20}$$

For high wave amplitudes this truncation cannot be rigorously justified. Other modes must be included and the calculation performed numerically. However, it appears that such truncations do give qualitatively correct solutions, at least for the Rossby wave case (Coaker, 1977). My aim is not to calculate accurate marginal stability curves (which would in the high amplitude case have little physical relevance because of the rapid onset of nonlinearity) but to illustrate the basic mechanisms of topographic instability. Hence I shall not pursue Coaker's more accurate approach involving Floquet theory. The dispersion relationship of (5.20) is:

$$(\sigma-\omega_{\mathbf{p}})(\sigma-\omega_{\mathbf{q}})(\sigma-\omega_{\mathbf{r}})-a_{pkq}a_{q-kp}|\tilde{\Phi}_{\mathbf{k}}|^2(k_\beta^2-q'^2)(k_\beta^2-p'^2)(\sigma-\omega_{\mathbf{r}})/p'^2q'^2$$

$$-a_{rkp}a_{p-kr}|\tilde{\Phi}_{\mathbf{k}}|^2(k_\beta^2-r'^2)(k_\beta^2-p'^2)(\sigma-\omega_{\mathbf{q}})/p'^2r'^2=0. \qquad (5.21)$$

The "high-wave amplitude" limit is obtained by neglecting all the ω's. This gives

$$\sigma^2=a_{pkq}a_{q-kp}|\tilde{\Phi}_{\mathbf{k}}|^2(k_\beta^2-q'^2)(k_\beta^2-p'^2)/p'^2q'^2$$

$$+a_{rkq}a_{p-kr}|\tilde{\Phi}_{\mathbf{k}}|^2(k_\beta^2-r'^2)(k_\beta^2-p'^2)/p'^2r'^2. \qquad (5.22)$$

This limit is valid near a resonance or when topography is high. The instability now is stationary, which interestingly, indicates the possibility of a modified, inherently nonlinear, stationary solution. In general, though, the instability is travelling.

Figure 1 plots contours of the instability for the purely barotropic problem (equivalent to $\lambda^2=0$). The axes of the figures are p_x and p_y and the lines are contours of $(-\sigma^2)$. Topography exists only at x-wavenumber unity (zero y-wavenumber) and σ^2 is then calculated for various values of k_β using (5.22). For $k<k_\beta$, form-drag instability is indeed reproduced: for small meridional wavenumber instability arises at the topographic scale and at the zonal average. Note that this does not justify the expansion procedure of Section 4. Rather, the consistency of the results justifies qualitatively the truncated spectral expansion. As soon as $k_\beta \geq k$, form-drag instability disappears (Figure 1b, c, d). The zonal instability is confined to smaller meridional scales and the non-zonal instability to values of $|p|$ larger than k_β. To illustrate the baroclinic problem let us consider only the geophysically interesting case $\lambda>|\mathbf{k}|$, and in particular let $\lambda/|\mathbf{k}|=3$. In Figure 2 are shown growth contours for a barotropic perturbation to a baroclinic basic state [i.e. the problem $(\tilde{\tau}_{\mathbf{k}}, \psi_{\mathbf{p}}, \tau_{\mathbf{q}}, \tau_{\mathbf{r}})$]. Again note the consistency with the previous form-drag instability: for all values of k_β for which $k_\beta^2<k^2+\lambda^2$ (i.e. $k_\beta<10^{1/2}$) the instability at low meridional wavenumbers is purely of the zonally-averaged barotropic flow. As soon as resonance is reached (Figure 2d) the form-drag instability at low p_y disappears. The subresonant instability does exist, but only at higher meridional

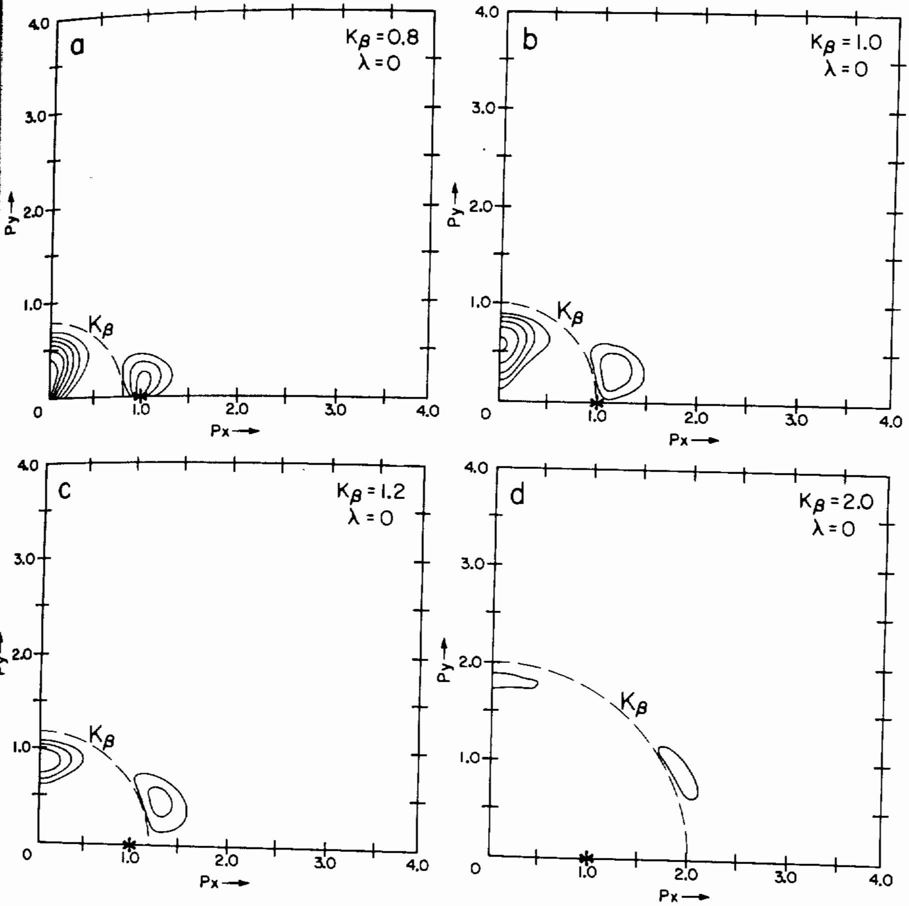

FIGURE 1 Contours of growth rate for the barotropic perturbation $\psi_{\mathbf{k}}$ on a barotropic basic state $\bar{\psi}_{\mathbf{k}}$ in the high amplitude limit for various values of k_β. Plotted are contours of $-\sigma^2$ where $-\sigma^2$ is given by (5.22) with $\lambda^2=0$ and $\bar{\psi}_{\mathbf{k}}$ of unit amplitude. Topography exists only at wavenumber $\mathbf{k}=(1,0)$, as marked by an asterisk. Only positive growth rates are drawn. Note that only for instability of small meridional wavenumber must the flow be superresonant ($k>k_\beta$), and that subresonant instability ($k<k_\beta$) exists at higher meridional wavenumbers.

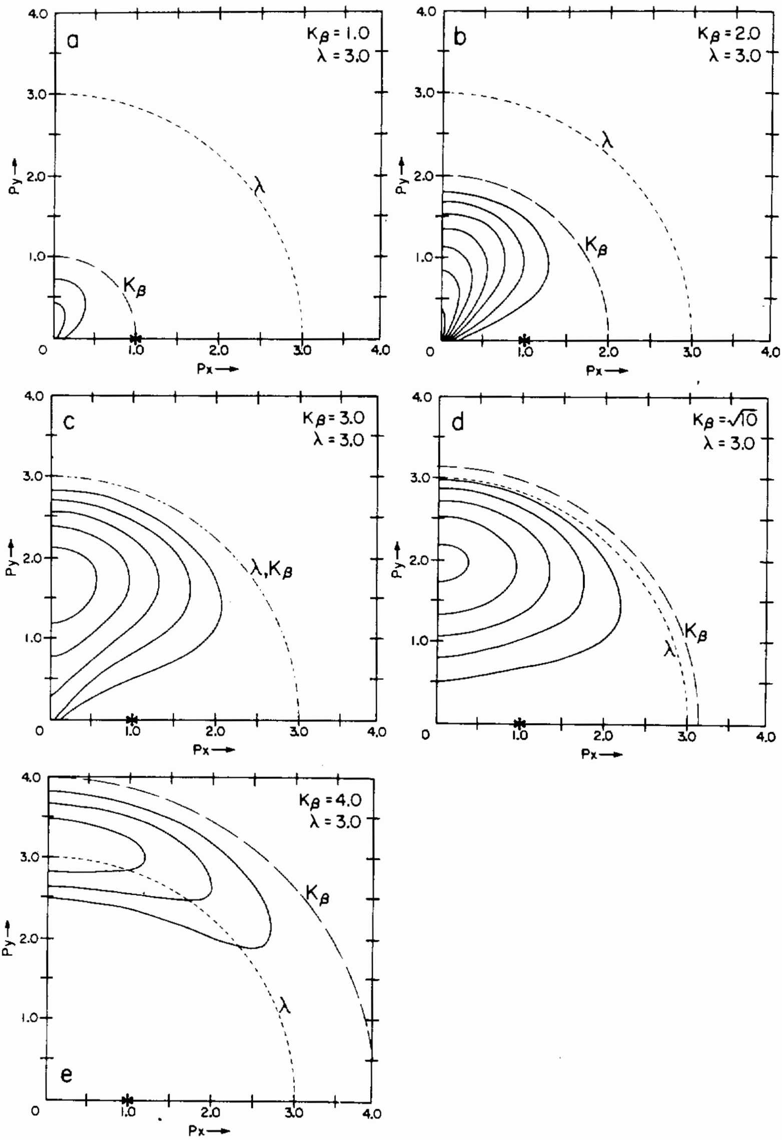

FIGURE 2 As for Figure 1 but now for barotropic perturbations ψ_p about a baroclinic basic state $\tilde{\tau}_{\mathbf{k}}$. Contour interval is ten times that of Figure 1. The values of k_β and λ $(=3)$ are marked with dashed lines. For resonant and subresonant zonal flow $(k_\beta^2 \geqq k^2 + \lambda^2)$ i.e. $(k_\beta^2 \geqq 10)$ the instability leaves the abscissa, as demanded by the asymptotic analysis of Section 4.

wavenumbers. The maximum growth rate occurs always at the zonal average (i.e. $p_x = 0$), the wavenumber increasing with (but always less than) k_β.

Figure 3 shows growth contours for a baroclinic perturbation growing from a baroclinic basic state [the problem $(\tau_k, \tau_p, \psi_q, \psi_r)$]. For superresonant flow ($k_\beta^2 < k^2 + \lambda^2$) the zonal scale of the instability is at the topographic scale, occurring when either ψ_q or ψ_r is a mode of the zonally averaged flow. For resonant and subresonant flow the form-drag instability disappears, the instability appearing in a band close to k_β.

6. SUMMARY AND CONCLUSIONS

This paper has primarily been concerned with the stationary flow set up by a uniform zonal current flowing over topography and the ensuing flow instabilities.

It was first demonstrated that for both one and two-layer flow, with arbitrary shear, exact fully nonlinear solutions exist. They are the same as the linear solution, i.e. that solution obtained by writing $\Psi = -Uy + \psi$ and ignoring terms in ψ^2. However, a simple functional relationship exists between the potential vorticity and the streamfunction which means that the nonlinear Jacobian terms $J(\psi, q)$ do vanish. Such solutions are the *only* nonlinear solutions if, far upstream or downstream away from the topographic influence, the flow is a uniform zonal current $\Psi_i = -U_i y$ and there are no closed streamfunction contours. For barotropic flow the relationship between potential vorticity and streamfunction is then $Q = -\beta\Psi/U$. More complicated, but similar, relationships exist in two-layer flow.

The remainder of the paper discussed the stability properties of this flow. If the topography is high enough, or if resonances exist in the system, such a flow will be unstable. Both subresonant and superresonant flow can be unstable. If the topography has little meridional variation, an asymptotic analysis can be used to display a form-drag instability. This is an instability involving the zonal flow, the forced stationary flow of topographic scale and a free mode also of topographic scale. The analysis is valid only for perturbations of large meridional scale, and only superresonant flow is unstable. Free Rossby waves and subresonant flow over topography do not exhibit

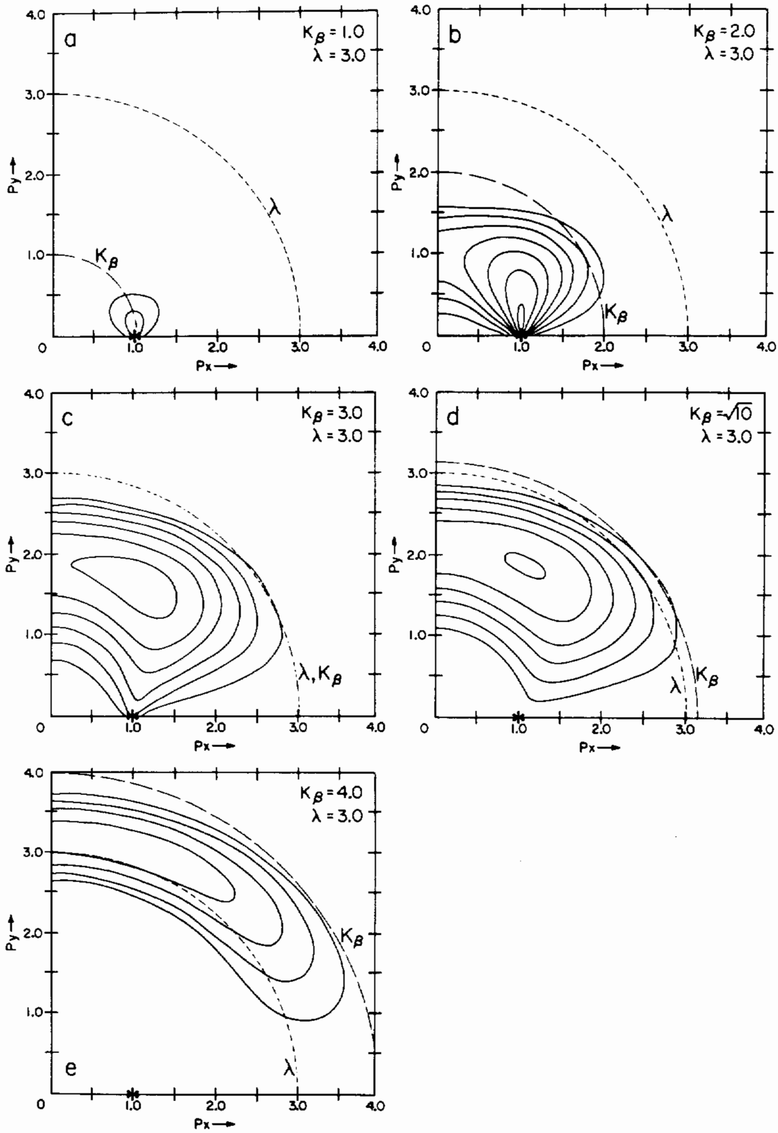

FIGURE 3 As for Figure 1 but now for baroclinic perturbations τ_p about a baroclinic basic state $\tilde{\tau}_k$. Contour interval is five times that of Figure 1. For resonant and subresonant flow the instability leaves the abscissa at the scale of the topography. There is never any instability of the zonally-averaged flow at small meridional wavenumber, consistent with the asymptotic analysis.

form-drag instability, but can nevertheless be unstable. For such flows an analysis based upon truncated Fourier expansion shows that the instability is a triad-interaction involving modes of higher meridional wavenumber, which are precluded from the asymptotic analysis. Form-drag instability may be considered a special kind of triad interaction involving interactions around an isosceles triangle in which the modes all have little meridional variation.

Instabilities were found which were both barotropic and baroclinic in nature. The two-layer problem may be put into a form formally equivalent to the one-layer problem, given suitable redefinitions of the scale of a wave. Form drag and other triad-instabilities exist in a precisely similar fashion. Baroclinic instability is shown to exist when the zonal flow, in the absence of topography, is manifestly stable. In particular, flow with *zero* mean shear can be baroclinically unstable. Integral constraints were found useful in giving necessary conditions for instability, for both the topographic problem and the conventional zonally symmetric instability problem. For the latter problem both the minimum shear and the high-wavenumber cut-off to instability, in two-level and continously stratified models, arise easily from the constraints. These are stronger results than can arise from a normal mode analysis, which in any case does not lead directly to simple stability criteria in the multi-level or continuously stratified case.

What is the relevance of topographic instability to stationary waves in the atmosphere and its modelling? Stationary, linear theory is often used to model the stationary response of the atmosphere to a given zonally-averaged zonal wind. To the extent that results from a β-plane theory can be applied to a spherical atmosphere, the theory suggest that the linear response may well be unstable. This would presumably cause the linear amplitude to be higher than that observed, because of the consequent transfer of energy from the unstable stationary asymmetric flow into transient flow, as is in fact observed (Holopainen, 1983) and modelled (Vallis and Roads, 1984). Storm-tracks are a manifestation of this, in which the transients grow, at least partially, under the influence of the stationary asymmetric flow.

Some blocking theories imagine blocks to involve the resonant response of flow over topography. In the original multiple-equilibrium theories the zonal flow plus two other modes are allowed. What are the limitations of this? Although the scale analysis of Section 2 indicates that near resonance a truncated spectral expansion is not justified, the neglect of other modes in the multi-equilibrium theories does not mean that the basic stationary states found in such models do not exist in higher resolution models. This is because the essential ingredient for multiple-equilibria is resonance, or more particularly that the form-drag on the zonal flow varies considerably and not monotonically with the zonal wind. However, such a neglect does mean that certain instabilities are not present in such models. Form-drag instability of the superresonant flow does exist. However the subresonant instability is lost. In reality there exist modes to which the subresonant flow is unstable with approximate growth rate $h\beta/\Delta$, where Δ is the distance from resonance $(\beta - Uk^2)$, and h is proportional to the mountain height. These modes are not of small meridional wave-number and so are also not included in Hart's analysis. The growth rate near resonance is order a few days, so very long-lived blocks cannot be explained by such a theory, or at least have not yet been so explained.

In the ocean topographic instability is possibly even more important than in the atmosphere since bottom features are so much higher. The exposition presented here is implicitly directed more at the atmosphere, since it was assumed that the relative depths of the two layers was the same. Another difference lies in the stratification, which gives the ocean a much smaller deformation radius and stronger baroclinic (rather than barotropic) effects. Nevertheless, the principles are much the same with the additional potential for a purely baroclinic triad $(\tilde{\tau}_k, \tau_p, \tau_q)$ due to unequal relative depths. The Antarctic Circumpolar Current is a potential location where such instabilities may be important, as indeed they evidently were in the simulations of McWilliams, Holland and Chow (1978). In their model the wind stress produces a mean shear which grows until unstable. The energy put in by the mechanical forcing can only be dissipated by eddies, hence the mean flow must be supercritical to some degree. With no topography the simulations produce a strong shear (≈ 0.2 m/s between upper and lowest layers). When topography is added, the shear is greatly reduced and indeed would be stable by conventional measures. However, eddies are still produced. Topographic instability is the likely candidate. If so, an energy spectra would reveal significant transient energy at topographic scales, larger than the deformation scale. Some zonal-averaged transients should be evident, also.

To end on a cautionary note, I shall mention some of the limitations of the analysis. A two-layer model with rigid-lid upper boundary condition cannot describe the possible important vertical radiation of energy in the atmosphere—it probably does a better job in the ocean, although then we would need to do the analysis with different equivalent depths. Also, much of the analysis above used highly idealized topographic forms, and was linear. The existence of resonance itself, on a sphere and with a realistic zonal wind, may be questionable. Finally, just because a given mode in a steady solution is unstable (say because it is resonantly forced and so of large amplitude) does not necessarily mean that the general form and structure of that solution is not maintained, if the whole system is forced. The instability, if weak, may simply modify the solution and the resonant mode still stand out.

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