On the Creation of Eddies in an Ideal Fluid.*

by

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(with two figures)

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ref. by Fellow Natanson.‡

In the 56-th Volume of the Wiedemann's Annales[1] Mr. Schütz shown that the famous theorem of v. Helmholtz[2] about the impossibility of generation of eddies in the ideal fluid, that is lacking the internal friction, via influence of conservative forces is valid only if certain additional conditions are satisfied.

Referring to the above mentioned work of Mr. Schütz, I would like to debate in this dissertation the following problem: what distribution of density and pressure must occur in the ideal fluid to allow a creation of vortices in the fluid, at a given instant? If such a distribution can be realized, with what speed and in which directions would develop the vortex filaments, at the moment of their creation?

I make it clear, that my intention is not to prove that the circumstances necessary for the generation of eddies in the ideal fluid via influence of conservative forces are physically imaginable and possible; my intention is to study these conditions in a rigorous and analytical way, and to derive a few appropriate theorems which due to their transparency could rather help in a prospective proof that a realization of such conditions is physically impossible.

Assuming solely the influence of conservative forces and using the general differential equations of hydrodynamics we obtain, according to Schütz and with a very simple manipulation, the following three equations for a fluid element, which at a given moment do not posses any rotational motion, yet:

$$\xi' = \frac{d\xi}{dt} = \frac{\partial}{\partial y}\left(\frac{1}{2\rho}\frac{\partial p}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{1}{2\rho}\frac{\partial p}{\partial y}\right), \tag{1}$$

$$\eta' = \frac{d\eta}{dt} = \frac{\partial}{\partial z}\left(\frac{1}{2\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial p}{\partial z}\right),$$

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‡It may mean that Wacław Natanson (mentioned in the Silberstein's biographical note) was a referee of the paper; at that time he was the Academy Fellow and Professor of physics at the Jagiellonian University in Cracow, known especially from his work on thermodynamics of irreversible processes. (M.Z.)

[1](that is in *Annalen der Physik* (M.Z.)) 1895., pages 144-147.

[2]*Wissenschaftl. Abhandlungen*, I., pages 101-134. "Ueb. Integrale d. hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen".

$$\zeta' = \frac{d\zeta}{dt} = \frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{1}{2\rho}\frac{\partial p}{\partial x}\right).$$

In these equations ξ, η, ζ represent components of (just emerging) vortical velocity, taken along coordinate axes x, y, z, at a moment t, at a point x, y, z inside the fluid; p is pressure, ρ is density at that point and at that instant.

Basing on these equations, we arrange to solve the problem formulated at the beginning of the dissertation and, especially, to obtain the solutions in a geometric, illustrative form.

After executing the differentiations from the right hand sides of equations (1), we get:

$$\begin{aligned}
\xi' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{y,z}\right), \\
\eta' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial z}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{z,x}\right), \\
\zeta' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{x,y}\right),
\end{aligned} \tag{2}$$

where $\left(\frac{p,\rho}{y,z}\right)$, and so on, are the abbreviations designating appropriate determinants.

Let us analyse surfaces:

1. of constant pressure:
$$p(x, y, z) = \text{ const.} \tag{3}$$

2. of constant density:
$$\rho(x, y, z) = \text{ const.}, \tag{4}$$

which, at the given instant, pass through the given point x, y, z. If these surfaces intersect each other, so that the point x, y, z lies on the intersection line, then the projections dx, dy, dz of the element of the intersection line ds, which we measure starting from the point x, y, z, must satisfy the following two equations:

$$\begin{aligned}
\frac{\partial p}{\partial x}\frac{dx}{dz} + \frac{\partial p}{\partial y}\frac{dy}{dz} &= -\frac{\partial p}{\partial z}, \\
\frac{\partial \rho}{\partial x}\frac{dx}{dz} + \frac{\partial \rho}{\partial y}\frac{dy}{dz} &= -\frac{\partial \rho}{\partial z}.
\end{aligned} \tag{5}$$

Solving these equations for dx/dz and dy/dz, we get:

$$dx : dy : dz = \left(\frac{p,\rho}{y,z}\right) : \left(\frac{p,\rho}{z,x}\right) : \left(\frac{p,\rho}{x,y}\right), \tag{6}$$

that is, according to (2):

$$dx : dy : dz = \xi' : \eta' : \zeta' \tag{7}$$

We have proved, therefore, the following theorem:

Theorem I: If in a given element of the ideal fluid influenced only via conservative forces a vortical motion is generated at certain instant, then the vortical axis of the fluid element coincides initially with the element of the intersection line of the surface of constant pressure with the surface of constant density to which the fluid element belongs at the moment; therefore, the developing vortex line initially coincides entirely with that intersection line.

This theorem does not assert, however, that the rotational motion must be always generated, if only the surfaces mentioned above intersect each other. We prove, however, that this is actually the case.

The mathematical expression for the resultant vortical acceleration ω' of the developing eddy is:

$$\omega' = \left(\xi'^{\,2} + \eta'^{\,2} + \zeta'^{\,2}\right)^{1/2} = \frac{1}{2\rho^2}\left[\left(\frac{p,\rho}{y,z}\right)^2 + \left(\frac{p,\rho}{z,x}\right)^2 + \left(\frac{p,\rho}{x,y}\right)^2\right]^{1/2}. \tag{8}$$

The components of the unit vectors n, ν normal to the surfaces p =const. and ρ =const., respectively, at the point x, y, z are:

$$a = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial x}, \quad b = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial y}, \quad c = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial z}, \tag{9}$$

and

$$\alpha = \frac{1}{\sqrt{R}}\frac{\partial \rho}{\partial x}, \quad \beta = \frac{1}{\sqrt{R}}\frac{\partial \rho}{\partial y}, \quad \gamma = \frac{1}{\sqrt{R}}\frac{\partial \rho}{\partial z}, \tag{10}$$

where, to be concise, we have defined:

$$P = \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2, \quad R = \left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial \rho}{\partial z}\right)^2. \tag{11}$$

If we define as positive these directions of n, ν for which pressure and density increase, then we need to choose the positive square roots in equations (9) and (10).

The angle θ between the positive directions of vectors n, ν is defined by the equation:

$$\cos\theta = \frac{1}{\sqrt{PR}}\left[\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial y} + \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial z}\right]. \tag{12}$$

On the other hand, the sum in brackets on the right hand side of equation (8) equals:

$$\begin{aligned}\omega'^{\,2}\cdot 4\rho^4 = {} & \left(\frac{\partial p}{\partial y}\right)^2\cdot\left(\frac{\partial \rho}{\partial z}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\cdot\left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\cdot\left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial x}\right)^2\cdot\left(\frac{\partial \rho}{\partial z}\right)^2 + \\ & + \left(\frac{\partial p}{\partial x}\right)^2\cdot\left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\cdot\left(\frac{\partial \rho}{\partial x}\right)^2 - 2\frac{\partial p}{\partial y}\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y}\frac{\partial \rho}{\partial z} - \\ & -2\frac{\partial p}{\partial z}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial z}\frac{\partial \rho}{\partial x} - 2\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\frac{\partial \rho}{\partial y} = \\ = {} & \left[\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\right]\left[\left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial \rho}{\partial z}\right)^2\right] - \\ & - \left[\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial y} + \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial z}\right]^2;\end{aligned} \tag{13}$$

linking this relation with equation (12) we get:

$$\cos^2\theta = 1 - \frac{4\rho^4\omega'^{\,2}}{PR}; \quad 4\rho^4\omega'^{\,2} = PR\sin^2\theta,$$

and therefore

$$\omega' = \frac{1}{2\rho^2}\sqrt{PR}\sin\theta, \tag{14}$$

or, because n, ν are the directions of the greatest growth of pressure and density (respectively), we have:

$$\omega' = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta. \tag{15}$$

We are arriving, therefore, at the following theorem:

Theorem II: The necessary and sufficient condition for generation of a vortical flow in an element of the ideal fluid influenced only by the conservative forces is that the surface of constant pressure and the surface of constant density, to which the given fluid element belongs and which until some specific instant coincided with each other ($\theta = 0$, or $\theta = \pi$), begin at that instant to intersect each other so that the analysed fluid element lies on the intersection line. The axis of the developing eddy coincides with the appropriate element of the intersection line: namely, the fluid element begins to rotate around this line element, from ν toward n (along the shorter path, see Fig. 1) with the vortical acceleration:

$$\omega' = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin(\nu, n) = \frac{1}{2\rho^2}V\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}. \tag{15}$$

The above result can also be expressed with other words:

The elements of the ideal fluid lacking the vortical motion do not start to rotate (under the influence of the conservative forces) if and only if pressure p can be expressed as a function of density ρ, only,[1] so that the respective surfaces of p and ρ coincide with each other, - or if ρ or p, or both of these quantities, do not depend on the location in the fluid (that is on x, y, z).

If, however, the surfaces of p and ρ are intersecting each other at some instant, the vortex line is immediately generated along their intersection, so that after time increment dt individual fluid elements of the vortex line gain appropriate vortical velocity:

$$d\omega = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta dt. \tag{16}$$

If swarms of surfaces p and ρ intersect each other, then instead of a single vortex line a whole vortex tube is formed, and its momentum after time dt can be calculated using equation (16). Let us consider an infinitesimally thin vortex tube, filling the channel between two surfaces of constant pressure: p and $p+\frac{\partial p}{\partial n}dn$, and two surfaces of constant density: ρ and $\rho+\frac{\partial \rho}{\partial \nu}d\nu$, which at some instant begin to intersect each other. Even without an assistance of the basic expressions for components ξ, η, ζ of the vortical velocity (via u, v, w) we can directly infer from equation (16) that the vortical momentum of the created vortex tube has one and this same value along the whole tube. This is because all intersections of the the analysed vortex tube are parallelograms, the sides of which are

$$a = dn \cdot \operatorname{cosec}\theta, \quad b = d\nu \cdot \operatorname{cosec}\theta, \tag{17}$$

[1]Pressure p depends generally on density ρ and temperature ϑ; if however surfaces of constant density are also the surfaces of constant temperature, we can express p as a function of density ρ, only. Surfaces p intersect surfaces ρ if and only if surfaces ϑ do not coincide with surfaces p nor with surfaces ρ. (See remarks in the final part of the dissertation).

and which intersect each other at the angle θ ($\theta = (n, \nu)$, as above); the cross-section of the tube equals, therefore

$$q = ab \sin\theta = dn \cdot d\nu \cdot \operatorname{cosec}\theta, \tag{18}$$

so that the momentum of the vortex tube after time dt from its generation is, according to (16):

$$\begin{aligned} q d\omega &= \frac{dt}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu} dn d\nu = \\ &= -\frac{dt}{2}\frac{\partial p}{\partial n} dn \frac{\partial}{\partial \nu}\left(\frac{1}{\rho}\right) d\nu; \end{aligned} \tag{19}$$

now, because the pressure incremet between surfaces of p as well as the density increment, and therefore the increment of the value of $\frac{1}{\rho}$, between surfaces of ρ are fixed along the whole channel, then the value of the momentum along the whole vortex tube is one and this same.

With time, however, the momentum of one and this same vortex tube undergoes continuous changes, namely as long as the surfaces of p and surfaces of ρ intersect each other, obviously even when the angle θ between them has a constant value. If that angle, however, equals zero or 180^o, that is, if beginning from some instant the surfaces p = const. and ρ =const. merge with each other, then the vortex tube preserves unchanged the value of momentum gained until this very moment; this steady state persists until the surfaces of p and ρ again begin to intersect each other along vortex lines forming a vortex tube. Using the general hydrodynamic equations we can easily analyse these theorems mathematically; to simplify the derivations we can analyse a two-dimensional flow of the fluid. The general equations (compare Schütz) expressing the relation between the components of the vortical velocity and their derivatives with respect of time, give for a special case - when we chose the plane yz as the "plane of the motion":

$$\begin{aligned} \eta = 0, \quad \zeta = 0, \quad \eta' = 0, \quad \zeta' = 0, \\ \omega' = \xi' = \frac{d\xi}{dt} = \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y}\right) - \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \end{aligned} \tag{20}$$ [1]

All surfaces of constant pressure and constant density are, in this case, cylindrical and normal to the plane yz, so that their normal vectors n, ν are everywhere parallel to this surface. Applying the normal vectors n, ν and angle $\theta = (n, \nu)$ to equation (20), we get:

$$\frac{d\xi}{dt} = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta - \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \tag{21}$$

This equation can be transformed according to our goals remembering that the spatial distribution of velocities of fluid elements is related to changes of density with time via the continuity equation, which in our case demands that:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w); \tag{22}$$

$\frac{\partial \rho}{\partial t}$ expresses the rate of the density change in a fixed element of space. Understanding that $\frac{d\rho}{dt}$ represents the change which takes place in the moving individual fluid element, instead of equation (22) we can write:

$$-\rho\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{d\rho}{dt}. \tag{23}$$

[1] v, w designate, as usually, velocities of fluid elements in directions of axes y and z.

After replacing in equation (21) the term $\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$ with the result of equation (23), we get:

$$\frac{d\xi}{dt}=\frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta+\xi\frac{1}{\rho}\frac{d\rho}{dt}, \tag{24}$$

hence, dividing both sides with ρ:

$$\frac{d}{dt}\left(\frac{\xi}{\rho}\right)=\frac{1}{2\rho^3}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta. \tag{25}$$

Because, however, density ρ is inversely proportional to the volume and, therefore, in the analysed two-dimensional case is inversely proportional to the cross-section area of the vortex tube so that ξ/ρ differs from the vortical momentum only via a constant factor, then the equation (25) asserts that the momentum of the vortex tube does not depend on time when and only when the surfaces of p and ρ generally coincide with each other, or at least they do not create any intersection line inside nor on the surface of the vortex tube. In the opposite case, the vortical momentum evolves with time according to the law which for the case of the two-dimensional flow is expressed, for instance, by equation (25).

Finally, I would like to show how an eddy motion is generated mechanically with the vortical acceleration expressed by equation (15), provided that the conditions expressed in Theorem II are satisfied. To do it, let us imagine an infinitesimally small parallelepiped (Fig. 2) with edges dn, $d\nu$, ds located inside the ideal fluid; n and ν are normal to the surfaces of constant p and constant ρ, respectively; ds is the element of the intersection line of these surfaces: ds is therefore perpendicular to the surface dn, $d\nu$. Let as consider the simplest case, in which the angle $\theta=(n,\nu)$ equals 90^o, so that the parallelepiped dn, $d\nu$, ds is rectangular. Let us divide the volume $dn\cdot d\nu\cdot ds$ into two equal parts, using the plane parallel to ds, dn. If the averaged density of the fluid contained in the parallelepiped is ρ, we can assume that its two parts have uniform densities $\rho-\frac{\partial\rho}{\partial\nu}\frac{d\nu}{4}$ and $\rho+\frac{\partial\rho}{\partial\nu}\frac{d\nu}{4}$, equal to the exact values of the density in their central points 1 and 2 (see Fig. 2). Then, the masses of these parts will be:

$$m_1=\left(\rho-\frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu\right)ds\,dn\frac{d\nu}{2}, \tag{26}$$

$$m_2=\left(\rho+\frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu\right)ds\,dn\frac{d\nu}{2}. \tag{27}$$

On every part of the parallelepiped a force is acting in the direction $-n$, equal:

$$N=\frac{\partial p}{\partial n}dn\cdot ds\frac{d\nu}{2}. \tag{28}$$

If we imagine that the analysed parts of the fluid solidified (every one individually), and that their total masses m_1, m_2 are located in their central points 1 and 2, then the forces N would induce the following accelerations of the material points 1 and 2, in the direction $-n$:

$$w_1=N:m_1=\frac{\partial p}{\partial n}:\left(\rho-\frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu\right), \tag{29}$$

$$w_2=N:m_2=\frac{\partial p}{\partial n}:\left(\rho+\frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu\right), \tag{30}$$

so that $|w_1| > |w_2|$. If we introduce:

$$w_1 = w_o + \frac{1}{2}(w_1 - w_2), \tag{31}$$

$$w_2 = w_o - \frac{1}{2}(w_1 - w_2), \tag{32}$$

the quantity w_o will be the acceleration of the translatory motion $[w_o = \frac{1}{2}(w_1 + w_2)]$ of the central point O of the whole parallelepiped in the direction $-n$, and the expression

$$\frac{1}{2}(w_1 - w_2) : \frac{1}{4}d\nu = 2(w_1 - w_2) : d\nu, \tag{33}$$

that is, according to (29) and (30), the expression

$$2(w_1 - w_2) : d\nu = 2\frac{\partial p}{\partial n}\left\{\frac{1}{\rho - \frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu} - \frac{1}{\rho + \frac{1}{4}\frac{\partial\rho}{\partial\nu}d\nu}\right\} : d\nu = \frac{1}{\rho^2}\frac{\partial p}{\partial n}\frac{\partial\rho}{\partial\nu} \tag{34}$$

will be the *angular* (rotational) acceleration of the set of the material points 1, 2 in the direction $\nu \to n$ around the the axis passing through the point O (which halves the distance $\overline{12}$) and being parallel to ds, therefore it will essentially be the doubled value of the *vortical* acceleration ω' in the point O, which was to be proven. We obtain the similar result in a general case, for which the surface of p forms whichever angle θ with the surface of ρ, in such a case the expression (34) is multiplied by $\sin\theta$, according to the theorem II.

I have derived and presented the above theorems only as mathematical corollaries, which arise from mechanical conditions of the ideal fluid influenced by the conservative forces. A completely different question, which we are not going to discuss here, is the question about appropriate physical conditions allowing the existence of actual lines of intersections of surfaces of p and surfaces of ρ. It seems that in advance, that is without conducting appropriate scientific research, it is only possible to say that if we generate in whatever way in a bulk of fluid having the above mentioned properties such a distribution of pressure and density, that the surfaces of p would intersect surfaces of ρ giving the origin to new eddies, and if we leave the fluid to its own, next, these forced distributions would change, after a very short time, in such a way that all the surfaces of p would again merge with the appropriate surfaces of ρ. Beginning from that moment, new eddies will not be created and the momentums of the vortex tubes which were generated during that short time will conserve their values.

[translation from Polish: Michał Ziemiański]