English translations of twenty-one of Ertel's papers on geophysical fluid dynamics

Wayne Schubert[1*], Eberhard Ruprecht[2], Rolf Hertenstein[1], Rosana Nieto Ferreira[3], Richard Taft[1], Christopher Rozoff[1], Paul Ciesielski[1], Hung-Chi Kuo[4]

[1]Department of Atmospheric Science, Colorado State University, USA
[2]Leibniz-Institut für Meereswissenschaften an der Universität Kiel
[3]GEST Center, University of Maryland, USA
[4]Department of Atmospheric Sciences, National Taiwan University, Taiwan

(Manuscript submitted January 20, 2004)

Abstract

This paper contains a collection of English translations of twenty-one of Hans Ertel's papers on geophysical fluid dynamics. The selected papers were originally published between 1942 and 1970 in either German or Spanish. This collection includes the four classic 1942 papers on vorticity and potential vorticity conservation principles and also papers on generalized conservation relations, hydrodynamic commutation formulas, Clebsch and Weber transformations, and isogons and isotachs in two-dimensional flows.

Zusammenfassung

Dieser Artikel ist eine Sammlung von 21 Veröffentlichungen zur Strömungsdynamik von Hans Ertel. Die ausgewählten Artikel erschienen ursprünglich in deutscher oder spanischer Sprache im Zeitraum von 1942 bis 1970. In der Sammlung enthalten sind die vier klassischen Artikel von 1942 über Vorticity und die Erhaltung potentieller Vorticity, sowie weitere Artikel zu allgemeinen Erhaltungssätzen, hydrodynamischen Vertauschungsrelationen, Clebsch- und Weber-Transformationen, und Isogonen und Isotachen in zweidimensiononalen Strömungen.

Preface

In view of the central role played by Ertel's potential vorticity in modern geophysical fluid dynamics, it may seem surprising that few of our fellow scientists have read Ertel's original works on the subject. This may be because most of these original papers were not written in English and were published in journals not easily accessible to many of us. It is in hopes of making Ertel's works more accessible that we provide the following set of translations.

The complete bibliography of Ertel's scientific contributions, published between 1929 and 1972, lists a staggering 272 papers. Most of the papers are short, containing a concise mathematical argument and a brief physical interpretation (figures are rare). The majority of the papers are on meteorology, but there also appear papers on oceanography, cosmology, atomic physics, volcanoes, fluvial erosion, etc. In selecting the papers for this special issue we have limited the topic to geophysical fluid dynamics and have been greatly aided by the collection of original German and Spanish articles reproduced in the five "Ertel volumes" edited by W. Schröder and H.-J. Treder, published in cooperation with the Interdivisional Commission on History of the International Association of Geomagnetism and Aeronomy (IAGA). The twenty-one translations presented here are as follows:

1. Ertel, H., 1942a: Ein neuer hydrodynamischer Wirbelsatz. Meteorol. Z. (Braunschweig), **59**(9), S. 277–281.

2. Ertel, H., 1942b: Ein neuer hydrodynamischer Erhaltungssatz. Die Naturwiss. (Berlin), **30**(36), S. 543–544.

3. Ertel, H., 1942c: Über hydrodynamische Wirbelsätze. Physik. Z. (Leipzig), **43**, S. 526–529.

4. Ertel, H., 1942d: Über das Verhältnis des neuen hydrodynamischen Wirbelsatzes zum Zirkulationssatz von V. Bjerknes. Meteorol. Z. (Braunschweig), **59**(12), S. 385–387.

5. Ertel, H., 1952: Über die physikalische Bedeutung von Funktionen, welche in der Clebsch-Transformation der hydrodynamischen Gleichungen auftreten. Sitz.-Ber. Dt. Ak. d. Wiss.

*Corresponding author: Wayne Schubert, Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado, USA, 80523, email: waynes@atmos.colostate.edu

1

Berlin, Klasse f. Math. u. allg. Naturwiss., 1952, No. 3, (Berlin 1952), S. 3–18.

6. ERTEL, H., 1955a: Kanonischer Algorithmus hydrodynamischer Wirbelgleichungen. Sitz.-Ber. Dt. Ak. d. Wiss. Berlin, Klasse f. Math. u. allg. Naturwiss., 1954, No. 4, (Berlin 1955), S. 3–10.

7. ERTEL, H., 1955b: Ein neues Wirbel-Theorem der Hydrodynamik. Sitz.-Ber. Dt. Ak. d. Wiss. Berlin, Klasse f. Math. u. allg. Naturwiss., 1954, No. 5, (Berlin 1955), S. 5–11.

8. ERTEL, H., 1956: Orthogonale Trajektoriensysteme in stationären ebenen Strömungsfeldern inkompressibler idealer Flüssigkeiten. Dt. Ak. d. Wissenschaften zu Berlin 1946–1956; Aus der Klasse für Mathematik, Physik und Technik, Berlin, S. 67–71.

9. ERTEL, H., 1957: Sobre una relación general entre la velocidad del viento y la intensidad del campo hidrodinámico en la atmósfera. Gerlands Beitr. z. Geophysik (Leipzig), **66**(4), p. 323–329.

10. ERTEL, H., 1960a: Teorema sobre invariantes sustanciales de la Hidrodinámica. Gerlands Beitr. z. Geophysik (Leipzig), **69**(5), p. 290–293.

11. ERTEL, H., 1960b: Relación entre la derivada individual y una cierta divergencia espacial en Hidrodinámica. Gerlands Beitr. z. Geophysik (Leipzig), **69**(6), p. 357–361.

12. ERTEL, H., 1961: Isoclinas e isótacas en corrientes potenciales bidimensionales de un flúido incompresible. Gerlands Beitr. z. Geophysik (Leipzig), **70**(1), p. 55–58.

13. ERTEL, H., 1963a: Analogía entre las ecuaciones del movimiento y las ecuaciones del torbellino en la Hidrodinámica. Gerlands Beitr. z. Geophysik (Leipzig), **72**(5), p. 312–314.

14. ERTEL, H., 1963b: Relaciones entre operaciones diferenciales del cálculo vectorial y paréntesis de Lagrange, con aplicación a la Hydrodinámica. Geofi sica pura e appl. (Milano), **55**, 1963/II, p. 119–122.

15. ERTEL, H., 1964: El número máximo de invariantes independientes en Hidrodinámica. Revista de Geofísica (Madrid), **XXIII**, Nums. 91/92, p. 121–124.

16. ERTEL, H., 1965a: Hydrodynamische Vertauschungs-Relationen. Acta Hydrophysica, **IX**(3), Berlin, S. 115–123.

17. ERTEL, H., 1965b: Kommutative Operatoren instationärer Strömungsfelder perfekter, piezotroper Flüssigkeiten. Monatsber. Dt. Ak. d. Wiss. Berlin, **7**(4), S. 296–298.

18. ERTEL, H., 1965c: Theorem über die unimodulare Transformation hydrodynamischer Numerierungskoordinaten. Gerlands Beitr. z. Geophysik (Leipzig), **74**(3), S. 255–260.

19. ERTEL, H., 1970a: Eine Relation zwischen kinematischen Parametern horizontaler Strömungsfelder in der Atmosphäre. Időjárás Budapest, **74**(1–2), S. 98–102.

20. ERTEL, H., 1970b: Spin und Deformationstensor im Zusammenhand mit Isogonen und Isotachen in ebenen Strömmungsfeldern. Acta Geodaet., Geophys. et Montanist. Acad. Sci. Hung., Tomus 5 (3–4), Budapest, pp. 383–387.

21. ERTEL, H., 1970c: Transformation der Differentialform der *Weber*schen hydrodynamischen Gleichungen unter Berücksichtigung der Erdrotation. Gerlands Beitr. z. Geophysik (Leipzig), **79**(6), p. 421–424.

Some of these papers are best studied in groups. For example, papers 1–4 present the results on potential vorticity with which we are so familiar. When these four papers were published in 1942, many readers probably thought that the new potential vorticity equation was very elegant, but with practical applications limited to tracer studies using special isentropic analyses. Later developments, however, would prove that Ertel's potential vorticity is much more useful than this simple initial understanding. The potential vorticity $\rho^{-1}(2\mathbf{\Omega} + \nabla \times \mathbf{v}) \cdot \nabla\theta$ is a combination of the wind and mass fields. For general nonbalanced fbws, knowledge of the potential vorticity field is not sufficient to determine both wind and mass. However, large-scale atmospheric fbws are quasi-static, and the wind and mass fields have mutually adjusted so that there exists some kind of approximate balance between them (e.g., geostrophic, gradient, or nonlinear balance). In such situations, the density ρ, the velocity $\mathbf{v}$, and the potential temperature θ can all be expressed in terms of the pressure p, so that the potential vorticity determines everything, if you can just invert it to find p. Thus, it is the concept of balance coupled with potential vorticity conservation which proves to be such a powerful tool in analyzing geophysical fbws. In addition, the transformation to different horizontal coordinates (e.g., geostrophic coordinates) or a different vertical coordinate (potential temperature) can often simplify the mathematics and lead to concise and elegant descriptions of the dynamics. In any case, the fundamental underlying notion is Ertel's potential vorticity (or an appropriate approximation to it); this is the motivation for our continuing interest in these classic 1942 papers.

Papers 6, 10, 11, and

22. ERTEL, H., and C.-G. ROSSBY, 1949: A new conservation theorem of hydrodynamics. Geofi sica Pura e Appl. (Milano), **14**, Fasc. 3–4, pp. 189–193.

can also be read as a group. The latter paper (22) is not included in the present collection since it was published in English (one of only two of Ertel's papers that were published in English). This group of papers deals with generalizations of the original 1942 results. In particular paper 6 provides a canonical algorithm from which the results of both the group 1–4 and paper 22 are easily obtained.

Finally, papers 8, 12, 19, and 20 can also be read as a group. Reflecting Ertel's interest in connecting dynamic meteorology with operational practice in synoptic meteorology, they deal with isogons and isotachs in two-dimensional flow and the use of natural coordinate systems.

In producing these translations we have attempted to preserve the formatting of the original papers as much as possible. References, for example, are cited in each paper with a style that matches that paper's original formatting, and the reference lists at the end of each paper are presented in their original untranslated forms. Certain symbols and notations, however, were updated to more modern conventions. For example, vector quantities have been represented using boldface symbols (rather than using an arrow symbol), and the "∇" symbol was used in place of the "grad" operator. Also, some formatting modifications were made in order to develop some coherency and consistency across the collection of papers as a whole (such as the formatting of section headings and equation numbers) and to deal with the two-column format of this journal (this affected the formatting of many of the longer formulas). Although Ertel's papers are largely free of typographical errors in the equations, we did detect a few obvious ones, and these have been corrected in the translations.

1   ERTEL (1942a): A new hydrodynamical vorticity equation

Relative to a rotating coordinate system, whose rotation is given by the constant rotation vector $\boldsymbol{\Omega}$, one can write the Eulerian hydrodynamical equation for an ideal compressible fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\tfrac{1}{2}\mathbf{v}^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + 2\boldsymbol{\Omega} \times \mathbf{v}$$
$$= -\nabla \Phi - \alpha \nabla p, \qquad (1.1)$$

together with the continuity equation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}. \qquad (1.2)$$

The foregoing symbols have the usual meaning ($\alpha = 1/\rho$ is the specific volume, Φ the potential of external forces).

The velocity $\mathbf{v}_e$ is related to the position vector $\mathbf{r}$ through

$$\mathbf{v}_e = \boldsymbol{\Omega} \times \mathbf{r}. \qquad (1.3)$$

The velocity of a mass element relative to an inertial system (non-rotating system) then becomes

$$\mathbf{v} + \mathbf{v}_e = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}. \qquad (1.4)$$

Since

$$\nabla \times (\boldsymbol{\Omega} \times \mathbf{r}) = 2\boldsymbol{\Omega}, \qquad (1.5)$$

then

$$\mathbf{Z} = \nabla \times (\mathbf{v} + \mathbf{v}_e) = \nabla \times \mathbf{v} + 2\boldsymbol{\Omega}, \qquad (1.6)$$

is the absolute vorticity, i.e., the vorticity relative to an inertial system, so that the Eulerian equation can be written

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\tfrac{1}{2}\mathbf{v}^2\right) - \mathbf{v} \times \mathbf{Z} = -\nabla \Phi - \alpha \nabla p. \qquad (1.7)$$

Using

$$\nabla \times (\alpha \nabla p) = -\nabla p \times \nabla \alpha \qquad (1.8)$$

and taking the curl of (1.7), we obtain

$$\frac{\partial}{\partial t} \nabla \times \mathbf{v} - \nabla \times (\mathbf{v} \times \mathbf{Z}) = \nabla p \times \nabla \alpha. \qquad (1.9)$$

Now consider $\psi = \psi(\mathbf{r}, t)$ to be a hydrodynamical invariant, i.e. a physical property which is conserved for every fluid element:

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = 0, \qquad (1.10)$$

as for example the potential temperature under adiabatic processes, or the polytropic temperature under polytropic conditions[1], or the total water content under conditions of no precipitation and without moisture input. In general, the invariant ψ does not have the same

value for all particles, but gives a field of $\psi = \text{constant}$ surfaces[2].

Scalar multiplication of (1.9) by $\nabla\psi$ yields

$$\nabla\psi \cdot \frac{\partial}{\partial t}\nabla \times \mathbf{v} - \nabla\psi \cdot \nabla \times (\mathbf{v} \times \mathbf{Z})$$
$$= (\nabla p \times \nabla\alpha) \cdot \nabla\psi, \qquad (1.11)$$

or

$$\nabla\psi \cdot \frac{\partial \mathbf{Z}}{\partial t} - \nabla\psi \cdot \nabla \times (\mathbf{v} \times \mathbf{Z}) = (\nabla p \times \nabla\alpha) \cdot \nabla\psi, \quad (1.12)$$

since from (1.6)

$$\frac{\partial \mathbf{Z}}{\partial t} = \frac{\partial}{\partial t}\nabla \times \mathbf{v}. \qquad (1.13)$$

Now, using

$$\nabla \cdot [\nabla\psi \times (\mathbf{v} \times \mathbf{Z})] = -\nabla\psi \cdot \nabla \times (\mathbf{v} \times \mathbf{Z}), \quad (1.14)$$

(1.12) becomes

$$\nabla\psi \cdot \frac{\partial \mathbf{Z}}{\partial t} + \nabla \cdot [\nabla\psi \times (\mathbf{v} \times \mathbf{Z})] = (\nabla p \times \nabla\alpha) \cdot \nabla\psi. \quad (1.15)$$

The vector algebra formula (triple vector product formula)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) \qquad (1.16)$$

results in

$$\nabla\psi \times (\mathbf{v} \times \mathbf{Z}) = \mathbf{v}(\mathbf{Z} \cdot \nabla\psi) - \mathbf{Z}(\mathbf{v} \cdot \nabla\psi), \quad (1.17)$$

which, along with

$$\mathbf{v} \cdot \nabla\psi = -\frac{\partial\psi}{\partial t},$$

allows us to write

$$\nabla\psi \times (\mathbf{v} \times \mathbf{Z}) = \mathbf{v}(\mathbf{Z} \cdot \nabla\psi) + \mathbf{Z}\frac{\partial\psi}{\partial t}. \qquad (1.18)$$

Taking the divergence, we obtain

$$\nabla \cdot [\nabla\psi \times (\mathbf{v} \times \mathbf{Z})] = \mathbf{v} \cdot \nabla(\mathbf{Z} \cdot \nabla\psi)$$
$$+ (\mathbf{Z} \cdot \nabla\psi)\nabla \cdot \mathbf{v} + \mathbf{Z} \cdot \frac{\partial}{\partial t}\nabla\psi \qquad (1.19)$$

since (1.5) and (1.6) imply

$$\nabla \cdot \mathbf{Z} = 0. \qquad (1.20)$$

Substitution of (1.19) into (1.15) results in

$$\nabla\psi \cdot \frac{\partial \mathbf{Z}}{\partial t} + \mathbf{v} \cdot \nabla(\mathbf{Z} \cdot \nabla\psi) + \mathbf{Z} \cdot \frac{\partial}{\partial t}\nabla\psi$$
$$+ (\mathbf{Z} \cdot \nabla\psi)\nabla \cdot \mathbf{v} = (\nabla p \times \nabla\alpha) \cdot \nabla\psi \qquad (1.21)$$

which, because of

$$\nabla\psi \cdot \frac{\partial \mathbf{Z}}{\partial t} + \mathbf{Z} \cdot \frac{\partial}{\partial t}\nabla\psi = \frac{\partial}{\partial t}(\mathbf{Z} \cdot \nabla\psi) \qquad (1.22)$$

and

$$\frac{\partial}{\partial t}(\mathbf{Z} \cdot \nabla\psi) + \mathbf{v} \cdot \nabla(\mathbf{Z} \cdot \nabla\psi) = \frac{d}{dt}(\mathbf{Z} \cdot \nabla\psi), \quad (1.23)$$

gives

$$\frac{d}{dt}(\mathbf{Z} \cdot \nabla\psi) + (\mathbf{Z} \cdot \nabla\psi)\nabla \cdot \mathbf{v} = (\nabla p \times \nabla\alpha) \cdot \nabla\psi. \quad (1.24)$$

Division by $\rho = 1/\alpha$ and use of the equation

$$\frac{1}{\rho}\nabla \cdot \mathbf{v} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{d\alpha}{dt} \qquad (1.25)$$

results in

$$\frac{d}{dt}(\alpha\mathbf{Z} \cdot \nabla\psi) = \alpha(\nabla p \times \nabla\alpha) \cdot \nabla\psi, \qquad (1.26)$$

an equation whose right hand side has a very simple physical meaning. To see this, consider the form

$$(\nabla\alpha' \times \nabla\beta') \cdot \nabla\gamma'.$$

In so far as there is no homogeneous relation $F(\alpha', \beta') = 0$ between the scalar functions α' and β', the surfaces $\alpha'(\mathbf{r}, t) = \text{constant}$ and $\beta'(\mathbf{r}, t) = \text{constant}$ form a solenoid system. If the third scalar function γ' is not of the form $\gamma' = \gamma'(\alpha', \beta')$, then this solenoid is divided into cells between the surfaces $\gamma'(\mathbf{r}, t) = \text{constant}$. Assume all three families of surfaces are given by

$$\ldots, \alpha' - 1, \alpha', \alpha' + 1, \ldots$$
$$\ldots, \beta' - 1, \beta', \beta' + 1, \ldots$$
$$\ldots, \gamma' - 1, \gamma', \gamma' + 1, \ldots$$

so that we have unit cells. Such a unit cell is shown in Fig. 1.1. It is formed by the intersections of the surfaces

$$\alpha' = \text{constant} \quad \text{and} \quad \alpha' + 1 = \text{constant}$$
$$\beta' = \text{constant} \quad \text{and} \quad \beta' + 1 = \text{constant}$$
$$\gamma' = \text{constant} \quad \text{and} \quad \gamma' + 1 = \text{constant}$$

The surface F in Fig. 1.1 is perpendicular to $\nabla\alpha' \times \nabla\beta'$ and it contains the directions of the gradient of α' (which is the direction $\mathbf{a}$) and of the gradient of β' (which is the direction $\mathbf{b}$). The direction of $\mathbf{c}$ is that of the gradient of γ' which is perpendicular to the surface of constant γ'. The vector $\mathbf{c}$ and the normal of F form the angle ν. The vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the surface F and
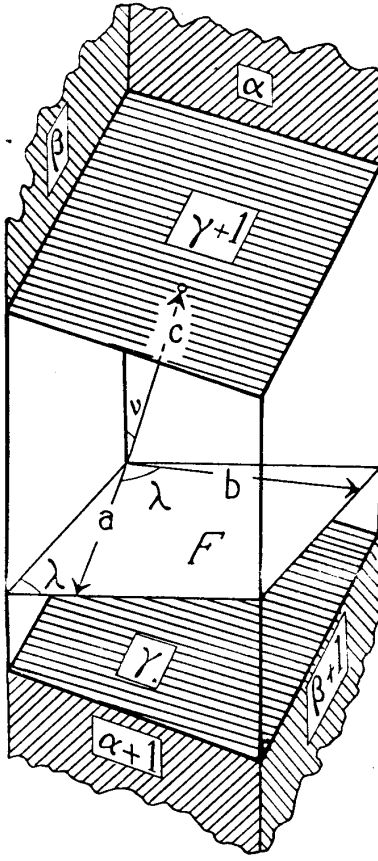
4

Fig. 1.1

form the angle λ. This is also (except for higher order terms) the angle between the surface $\alpha' + 1$ and the surface β' (and between the surface α' and the surface $\beta' + 1$), because $\mathbf{a} \perp \alpha'$ ($\mathbf{a} \perp \alpha' + 1$) and $\mathbf{b} \perp \beta'$ ($\mathbf{b} \perp \beta' + 1$). The length of $\mathbf{c}$ is the height of the unit cell, because $\mathbf{c}$ gives the distance between the surfaces of $\gamma' + 1$ and γ'. The surface area of F is

$$F = \frac{ab}{\sin \lambda} \qquad (1.27)$$

because $a/\sin \lambda$ is the side of F to which $\mathbf{b}$ is perpendicular. Because the perpendicular to γ' and the perpendicular to F form the angle ν, the projection of F onto γ' is given by

$$\text{projection of } F = \frac{F}{\cos \nu}. \qquad (1.28)$$

The volume V of the unit cell is $c(F/\cos \nu)$, or

$$V = \frac{abc}{\sin \lambda \cos \nu}. \qquad (1.29)$$

Now,

$$(\nabla \alpha' \times \nabla \beta') \cdot \nabla \gamma'$$
$$= |\nabla \alpha'||\nabla \beta'| \sin \lambda |\nabla \gamma'| \cos \nu, \qquad (1.30)$$

and since

$$\left. \begin{array}{l} |\nabla \alpha'| = \dfrac{(\alpha' + 1) - \alpha'}{a} = \dfrac{1}{a} \\[2mm] |\nabla \beta'| = \dfrac{(\beta' + 1) - \beta'}{b} = \dfrac{1}{b} \\[2mm] |\nabla \gamma'| = \dfrac{(\gamma' + 1) - \gamma'}{c} = \dfrac{1}{c} \end{array} \right\}, \qquad (1.31)$$

we have

$$(\nabla \alpha' \times \nabla \beta') \cdot \nabla \gamma' = \frac{\sin \lambda \cos \nu}{abc}, \qquad (1.32)$$

which together with (1.29) gives

$$(\nabla \alpha' \times \nabla \beta') \cdot \nabla \gamma' = \frac{1}{V}, \qquad (1.33)$$

which is the inverse volume of an $(\alpha', \beta', \gamma')$ unit cell or the number of $(\alpha', \beta', \gamma')$ unit cells in a unit volume. Thus,

$$\alpha(\nabla \alpha' \times \nabla \beta') \cdot \nabla \gamma' = \frac{\alpha}{V} = N(\alpha', \beta', \gamma'), \quad (1.34)$$

the number of $(\alpha', \beta', \gamma')$ unit cells in the specific volume, i.e. in the volume occupied by a unit mass of fluid. Then,

$$\boxed{\alpha(\nabla p \times \nabla \alpha) \cdot \nabla \psi = N(p, \alpha, \psi)} \qquad (1.35)$$

5

the number of (p, α, ψ) unit cells in the specific volume or the number of (p, α, ψ) unit cells per unit mass. The sign of $N(p, \alpha, \psi)$ is positive if the vectors ∇p, $\nabla \alpha$ and $\nabla \psi$ are cyclic; otherwise it is negative. Thus, we have

$$(\nabla p \times \nabla \alpha) \cdot \nabla \psi = (\nabla \psi \times \nabla p) \cdot \nabla \alpha$$
$$= (\nabla \alpha \times \nabla \psi) \cdot \nabla p \qquad (1.36)$$

or

$$\boxed{N(p, \alpha, \psi) = N(\psi, p, \alpha) = N(\alpha, \psi, p),} \qquad (1.37)$$

that is, the sign of $N(p, \alpha, \psi)$ remains unchanged during cyclic changes. Because of

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \qquad (1.38)$$

during non-cyclic changes, we have

$$\boxed{N(p, \alpha, \psi) = -N(p, \psi, \alpha) = -N(\psi, \alpha, p).}$$
$$(1.39)$$

Equation (1.26) can now be written as

$$\boxed{\begin{aligned}\frac{d}{dt}(\alpha \mathbf{Z} \cdot \nabla \psi) &= \frac{d}{dt}\{\alpha(\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \cdot \nabla \psi\} \\ &= N(p, \alpha, \psi),\end{aligned}}$$
$$(1.40)$$

which is the analytical form of the following vorticity theorem:

> *The individual time change of the specific volume multiplied by the scalar product of the absolute vorticity vector and the gradient of a conservative property of every fluid particle is equal to the number of (p, α, ψ) unit cells contained in the specific volume α.*

An important special case of the foregoing new vorticity theorem occurs when the conservative property ψ can be written as a function of p and α,

$$\psi = \psi(p, \alpha), \qquad (1.41)$$

for example, in the case when ψ is the polytropic temperature (see R. Emden). Then we have

$$\nabla \psi = \frac{\partial \psi}{\partial p}\nabla p + \frac{\partial \psi}{\partial \alpha}\nabla \alpha, \qquad (1.42)$$

and, as a consequence,

$$\begin{aligned}N(p, \alpha, \psi) &= \alpha \frac{\partial \psi}{\partial p}(\nabla p \times \nabla \alpha) \cdot \nabla p \\ &\quad + \alpha \frac{\partial \psi}{\partial \alpha}(\nabla p \times \nabla \alpha) \cdot \nabla \alpha \\ &= 0\end{aligned} \qquad (1.43)$$

and

$$(\nabla p \times \nabla \alpha) \cdot \nabla p = (\nabla p \times \nabla \alpha) \cdot \nabla \alpha = 0. \qquad (1.44)$$

The volume V of a (p, α, ψ) unit cell becomes infinite and the cells degenerate into solenoids; the (p, α) solenoid, the (p, ψ) solenoid, and the (ψ, α) solenoid have the same direction at each point, a direction which, however, can be different for different points.

The vorticity theorem (1.40) then becomes a hydrodynamical conservation theorem

$$\boxed{\frac{d}{dt}\{\alpha(\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \cdot \nabla \psi\} = 0,} \qquad (1.45)$$

the derivation of which I have recently given in another way[3]. The adiabatic motion in the atmosphere (as a special case of the polytropic case) satisfies

$$\boxed{\frac{d}{dt}\{\alpha(\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \cdot \nabla \theta\} = 0.} \qquad (1.46)$$

Here,

$$\theta = T\left(\frac{1000}{p}\right)^{\kappa} = \frac{1000^{\kappa}}{R}p^{1-\kappa}\alpha \qquad (1.47)$$

is the potential temperature (R = gas constant, $\kappa = R/c_p$). The adiabatic special case of the new vorticity theorem, that is (1.46), may prove to be a very useful tool for isentropic analysis (C. G. Rossby).

Here let us discuss as an application of (1.46), the beginning of an adiabatic vertical motion along the vertical axis of expansion or contraction of a volume of air. If the positive z axis is upward and if (x, y, z) is a right-hand cartesian system, the horizontal (x, y) surface touches the earth's surface at latitude φ and

$$\mathbf{k} \cdot \nabla \times \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta \qquad (1.48)$$

($\zeta > 0$ for cyclonic motion and $\zeta < 0$ for anticyclonic motion) and

$$\mathbf{k} \cdot \mathbf{\Omega} = \Omega \sin \varphi > 0 \quad \text{(for the northern} \atop \text{hemisphere).} \qquad (1.49)$$

The conservation theorem (1.46) gives

$$\frac{d}{dt}\left\{\alpha(\zeta + 2\Omega \sin \varphi)\frac{\partial \theta}{\partial z}\right\} = 0, \qquad (1.50)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + w\frac{\partial}{\partial z} \qquad (1.51)$$

6

because $u = v = 0$ at the vertical axis (however, $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y \neq 0$). From (1.50) we obtain

$$\frac{d}{dt}(\zeta + 2\Omega \sin \varphi) =$$
$$- (\zeta + 2\Omega \sin \varphi) \left(\alpha \frac{\partial \theta}{\partial z} \right)^{-1} \frac{d}{dt} \left(\alpha \frac{\partial \theta}{\partial z} \right). \quad (1.52)$$

At the initial time, we assume $\zeta = 0$; also in this case, $d(2\Omega \sin \varphi)/dt = 0$, so that from (1.52)

$$\frac{d\zeta}{dt} = -2\Omega \sin \varphi \left(\alpha \frac{\partial \theta}{\partial z} \right)^{-1} \frac{d}{dt} \left(\alpha \frac{\partial \theta}{\partial z} \right). \quad (1.53)$$

Now[4],

$$\left(\alpha \frac{\partial \theta}{\partial z} \right)^{-1} \frac{d}{dt} \left(\alpha \frac{\partial \theta}{\partial z} \right) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_\theta \quad (1.54)$$

where $(\partial u/\partial x + \partial v/\partial y)_\theta$ is the divergence on an isentropic surface. Thus, from (1.53) and (1.54) it follows that

$$\frac{d\zeta}{dt} = -2\Omega \sin \varphi \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_\theta. \quad (1.55)$$

With this we get for convergence, $\partial u/\partial x + \partial v/\partial y < 0$, the development of cyclonic vorticity, while for divergence, $\partial u/\partial x + \partial v/\partial y > 0$, the development of anticyclonic vorticity. Since $\alpha(\partial \theta/\partial z) = (1/\rho)(\partial \theta/\partial z)$ is a measure of the vertical stability of an air mass, we can develop the following interpretation: In an air mass without relative vorticity, anticyclonic (cyclonic) vorticity will be developed by adiabatic vertical motion if the stability $\alpha \partial \theta/\partial z$ increases (decreases).

The vorticity equation (1.40) can also be written in integral form. Multiplying by the mass element $\rho d\tau$ ($d\tau$ = volume element) and integrating over the total mass $m = \iiint \rho d\tau$, we obtain[5]

$$\boxed{\begin{aligned} \iiint \rho \frac{d}{dt}(\alpha \mathbf{Z} \cdot \nabla \psi) \, d\tau &= \frac{d}{dt} \iiint \mathbf{Z} \cdot \nabla \psi \, d\tau \\ &= N_m(p, \alpha, \psi), \end{aligned}} \quad (1.56)$$

where $N_m(p, \alpha, \psi)$ is the number of (p, α, ψ) unit cells contained in the mass m. Since $\mathbf{Z} \cdot \nabla \psi = \nabla \cdot (\psi \mathbf{Z})$, Gauss' theorem allows us to write (1.56) as

$$\boxed{\frac{d}{dt} \iint Z_n \psi \, d\sigma = N_m(p, \alpha, \psi),} \quad (1.56a)$$

where $d\sigma$ is an area element of the closed surface and n denotes the outward normal.

With a non-conservative quantity ψ^* (i.e., $d\psi^*/dt \neq 0$) one obtains the new vorticity theorem[6] in its general form since one writes $d\psi^*/dt - \partial \psi^*/\partial t$ instead of $\mathbf{v} \cdot \nabla \psi$ from (1.17) on. This generalized vorticity theorem takes the form

$$\boxed{\frac{d}{dt}(\alpha \mathbf{Z} \cdot \nabla \psi^*) = N(p, \alpha, \psi^*) + \alpha \mathbf{Z} \cdot \nabla \frac{d\psi^*}{dt},} \quad (1.57)$$

which can be expressed as follows:

In an ideal fluid, the individual time change of the specific volume multiplied by the scalar product of the absolute vorticity vector and the gradient of a field function is equal to the number of (p, α, ψ^*) unit cells plus the gradient of $d\psi^*/dt$ projected onto the vector $\alpha \mathbf{Z}$.

From (1.57) one obtains, for example, the generalized Helmholtz vorticity equation as a special case if one gives instead of ψ^* the components of the position vector $\mathbf{r} = (x, y, z)$.

The integral form of (1.57) is

$$\boxed{\begin{aligned} \frac{d}{dt} \iint Z_n \psi^* d\sigma &= N_m(p, \alpha, \psi) \\ &+ \iint Z_n \frac{d\psi^*}{dt} d\sigma, \end{aligned}} \quad (1.58)$$

where $d\sigma$ is an area element (n = outward normal) of the volume τ. $N_m(p, \alpha, \psi^*)$ has the same meaning as above; the derivation makes use of $\mathbf{Z} \cdot \nabla (d\psi^*/dt) = \nabla \cdot [\mathbf{Z} (d\psi^*/dt)]$.

References

[1] R. Emden, Thermodynamik der Himmelskörper. Enzykl. Math. Wiss., VI, 2, 24. Leipzig-Berlin 1926, S. 389.

[2] V. Bjerknes u. Mitarbeiter, Physikalische Hydrodynamik. Berlin 1933, S. 1–8.

[3] H. Ertel, Ein neuer hydrodynamischer Erhaltungssatz. Die Naturwissenschaften, Berlin 1942.

[4] H. Ertel, Über die individuelle Änderung des vertikalen Temperaturgradienten in der Atmosphäre. Met. Z. 1941, S. 223–236.

[5] H. Ertel, Methoden und Probleme der Dynamischen Meteorologie. Berlin 1938, S. 21.

[6] H. Ertel, Über hydrodynamische Wirbelgleichungen. Phys. Z. 1942.

2 ERTEL (1942b): A new hydrodynamical conservation principle

In a compressible fluid whose particles move polytropically, the specific volume α can be expressed as a function of the pressure p and the "polytropic temperature" $\Theta_s = T \left(\rho_0/\rho\right)^{1/s}$ ($s = \left(c_v - \beta\right)/\left(c_p - c_v\right)$ is the polytropic order, $\beta = dQ/dT$ the constant polytropic specific heat capacity, $\rho = 1/\alpha$ the density, and $\rho_0 = 1$ g/cm^3)[1]:

$$\alpha = \frac{1}{\rho} = \psi(p, \Theta_s). \tag{2.1}$$

In general, Θ_s varies from particle to particle but remains constant in time for a given particle:

$$\frac{d\Theta_s}{dt} = \frac{\partial \Theta_s}{\partial t} + \mathbf{v} \cdot \nabla \Theta_s = \frac{\partial \Theta_s}{\partial t} + v_k \frac{\partial \Theta_s}{\partial x_k} = 0, \tag{2.2}$$

the last part of which makes use of the usual summation convention. With respect to a rotating rectangular cartesian coordinate system (x_i, $i = 1, 2, 3$) whose rotation is described by the constant rotation vector $\mathbf{\Omega}$, the hydrodynamical vorticity equation[2] can be written

$$\frac{d}{dt}\left(\frac{\xi_i}{\rho}\right) - \left(\frac{\xi_k}{\rho}\right)\frac{\partial v_i}{\partial x_k} = -\frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\nabla p\right), \tag{2.3}$$

where

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} + 2\mathbf{\Omega} \tag{2.4}$$

denotes the absolute (relative to an inertial system) vorticity vector and where $\mathbf{v} = v_i$ ($i = 1, 2, 3$) denotes the wind relative to the rotating x_i system.

On account of (2.1),

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \frac{\partial \psi}{\partial \Theta_s}\nabla\Theta_s \times \nabla p$$

or in component form

$$\left[\nabla \times \left(\frac{1}{\rho}\nabla p\right)\right]_i = \frac{\partial \psi}{\partial \Theta_s}\epsilon_{ijk}\frac{\partial \Theta_s}{\partial x_j}\frac{\partial p}{\partial x_k},$$

where ϵ_{ijk} denotes the following tensor:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for a cyclic sequence of indices} \\ -1 & \text{for a non-cyclic sequence of indices} \\ 0 & \text{if any two indices agree} \end{cases}$$

The vorticity equation (2.3) can now be written ($\boldsymbol{\xi} = \xi_i$, $i = 1, 2, 3$)

$$\frac{d}{dt}(\alpha\xi_i) - (\alpha\xi_k)\frac{\partial v_i}{\partial x_k} = -\alpha\frac{\partial \psi}{\partial \Theta_s}\epsilon_{ijk}\frac{\partial \Theta_s}{\partial x_j}\frac{\partial p}{\partial x_k}. \tag{2.5}$$

Multiplying by $\partial\Theta_s/\partial x_r$ and making use of

$$\epsilon_{ijk}\frac{\partial \Theta_s}{\partial x_i}\frac{\partial \Theta_s}{\partial x_j}\frac{\partial p}{\partial x_k} = 0,$$

it follows from (2.5) that

$$\frac{\partial \Theta_s}{\partial x_i}\frac{d}{dt}(\alpha\xi_i) - (\alpha\xi_k)\frac{\partial v_i}{\partial x_k}\frac{\partial \Theta_s}{\partial x_i} = 0, \tag{2.6}$$

or in another form

$$\frac{d}{dt}\left(\alpha\frac{\partial \Theta_s}{\partial x_i}\xi_i\right) - (\alpha\xi_i)\frac{d}{dt}\left(\frac{\partial \Theta_s}{\partial x_i}\right) \\ - (\alpha\xi_k)\frac{\partial v_i}{\partial x_k}\frac{\partial \Theta_s}{\partial x_i} = 0. \tag{2.7}$$

From (2.2) we now obtain

$$\frac{\partial}{\partial x_i}\left(\frac{d\Theta_s}{dt}\right) = \frac{d}{dt}\left(\frac{\partial \Theta_s}{\partial x_i}\right) + \frac{\partial \Theta_s}{\partial x_k}\frac{\partial v_k}{\partial x_i} = 0, \tag{2.8}$$

so that

$$\xi_i\frac{d}{dt}\left(\frac{\partial \Theta_s}{\partial x_i}\right) = -\xi_i\frac{\partial \Theta_s}{\partial x_k}\frac{\partial v_k}{\partial x_i} = -\xi_k\frac{\partial \Theta_s}{\partial x_i}\frac{\partial v_i}{\partial x_k}, \tag{2.9}$$

and the substitution of (2.9) into (2.7) leads to the following conservation principle:

$$\boxed{\frac{d}{dt}\left(\alpha\xi_i\frac{\partial \Theta_s}{\partial x_i}\right) = \frac{d}{dt}\left\{\alpha\left(\nabla \times \mathbf{v} + 2\mathbf{\Omega}\right)_n\nabla_n\Theta_s\right\} \\ = 0,}$$

$$\tag{2.10}$$

where n denotes the direction normal to the planes of constant polytropic temperature and toward increasing Θ_s. Equation (2.10) is the analytical version of the following conservation principle[3]:

> *In a fluid which moves polytropically, the quantity formed by multiplying the specific volume by the scalar product of the absolute vorticity vector and the gradient of the polytropic temperature is conserved following individual particles.*

For $s = c_v/(c_p - c_v)$, the polytropic temperature reduces to the potential temperature

$$\theta = T\left(\frac{P}{p}\right)^\kappa \tag{2.11}$$

(P = reference pressure, e.g. 1000 mb, and $\kappa = R/c_p$) and (2.2) becomes

$$\frac{d\theta}{dt} = 0.$$

The conservation principle then becomes

$$\frac{d}{dt}\left\{\alpha\left(\nabla\times\mathbf{v}+2\mathbf{\Omega}\right)_n\nabla_n\theta\right\}=0, \qquad (2.12)$$

which in meteorology is connected to the isentropic analysis work of C. G. Rossby.

References

[1] R. EMDEN, Thermodynamik der Himmelskörper.

Enzykl. Math. Wiss., VI, 2, 24, S. 389. Berlin-Leipzig 1926.

[2] W. VOIGT, Kompendium der theoretischen Physik I, 270.–Leipzig 1895.

[3] The conservation principle given here can be regarded as a special case of the new hydrodynamical vorticity equation which I have proven in another paper. (Meteorol. Z. 1942).

3   ERTEL (1942c): On hydrodynamical vorticity equations

I. Introduction

Adopting the usual summation convention (according to which every index occurring twice in a single term is summed from 1 to 3) the vorticity equation for an ideal compressible fluid reads as follows:

$$\frac{d}{dt}\left(\alpha\zeta_i\right)-\left(\alpha\zeta_k\right)\frac{\partial w_i}{\partial z_k}=\epsilon_{ijk}\alpha\frac{\partial p}{\partial z_j}\frac{\partial\alpha}{\partial z_k}, \qquad (3.1)$$

where $\alpha=\rho^{-1}$ is the specific volume, p the pressure, $z_i(i=1,2,3)$ the rectangular cartesian (inertial) coordinate system, $w_i(i=1,2,3)$ the wind components, $\zeta_i=\epsilon_{ijk}\left(\partial w_k/\partial z_j\right)=(\nabla\times\mathbf{w})_i$, and

$$\epsilon_{ijk}=\begin{cases}1 & \text{for a cyclic sequence of indices,}\\-1 & \text{for a non-cyclic sequence of indices,}\\0 & \text{if any two indices agree.}\end{cases}$$

Through a generalization of (3.1) we shall now derive a new vorticity equation.

II. General formulation of hydrodynamical vorticity equations by means of a commutator relation

In order to treat the most general case we refer not to the inertial system as in (3.1) but rather to a rotating rectangular coordinate system x_i, whose rotation is described by the constant rotation vector $f_i(i=1,2,3)$; the vorticity equations then take the form

$$\frac{d}{dt}\left(\alpha\xi_i\right)-\left(\alpha\xi_k\right)\frac{\partial v_i}{\partial x_k}=\epsilon_{ijk}\alpha\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k} \qquad (3.2)$$

with the individual (material) derivative operator

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial x_k} \qquad (3.3)$$

and the absolute vorticity

$$\xi_i=\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}+2f_i, \qquad (3.4)$$

in which $v_i(i=1,2,3)$ is the wind component and $\epsilon_{ijk}\left(\partial v_k/\partial x_j\right)$ is the vorticity relative to the rotating x_i-system. The existence of a potential for other forces is implicit in (3.2).

Scalar multiplication of (3.2) by the gradient $\partial\psi/\partial x_i$ of a hydrodynamical field function ψ (scalar, vector or tensor component) leads easily to the form

$$\frac{d}{dt}\left(\alpha\xi_i\frac{\partial\psi}{\partial x_i}\right)-\left(\alpha\xi_i\right)\frac{d}{dt}\left(\frac{\partial\psi}{\partial x_i}\right)-\left(\alpha\xi_k\right)\frac{\partial\psi}{\partial x_i}\frac{\partial v_i}{\partial x_k}$$
$$=\alpha\epsilon_{ijk}\frac{\partial\psi}{\partial x_i}\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k},$$

which, through the exchange of the summation indices in the third term on the left side, can be written

$$\frac{d}{dt}\left(\alpha\xi_i\frac{\partial\psi}{\partial x_i}\right)-\left(\alpha\xi_i\right)\left\{\frac{d}{dt}\left(\frac{\partial\psi}{\partial x_i}\right)+\frac{\partial\psi}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right\}$$
$$=\alpha\epsilon_{ijk}\frac{\partial\psi}{\partial x_i}\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k}. \qquad (3.5)$$

Now, from (3.3), we have

$$\frac{d}{dt}\left(\frac{\partial\psi}{\partial x_i}\right)=\frac{\partial^2\psi}{\partial x_i\partial t}+v_k\frac{\partial^2\psi}{\partial x_i\partial x_k}$$
$$=\frac{\partial}{\partial x_i}\left(\frac{d\psi}{dt}\right)-\frac{\partial\psi}{\partial x_k}\frac{\partial v_k}{\partial x_i},$$

with which (3.5) can be reduced to

$$\frac{d}{dt}\left(\alpha\xi_i\frac{\partial\psi}{\partial x_i}\right)-\left(\alpha\xi_i\frac{\partial}{\partial x_i}\right)\frac{d\psi}{dt}$$
$$=\alpha\epsilon_{ijk}\frac{\partial\psi}{\partial x_i}\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k}, \qquad (3.6)$$

whose right hand side has the following simple meaning: Assume that the surfaces $p =$ const, $\alpha =$ const and $\psi =$ const (with scalar value differences of 1) are constructed throughout space; each set of surfaces divides space into unit layers. As long as there is no homogeneous relation $F(p,\alpha) = 0$ (V. Bjerknes[1]) between p and α, the intersections of the p and α unit layers form (p,α) unit solenoids. The (p,α) unit solenoids are intersected by the ψ unit layers, as long as ψ is not of the form $\psi = \psi(p,\alpha)$. In this way a field of (p,α,ψ) unit cells is formed. We can then interpret

$$\epsilon_{ijk}\frac{\partial\psi}{\partial x_i}\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k} = (\nabla p \times \nabla\alpha)\cdot\nabla\psi = \pm\frac{1}{V} \quad (3.7)$$

as the reciprocal volume V of a (p,α,ψ) unit cell or the number of (p,α,ψ) unit cells in a unit volume. The proof is accomplished by integration of $(\nabla p \times \nabla\alpha)\cdot\nabla\psi = \nabla\cdot(\psi\nabla p \times \nabla\alpha)$ over the volume of a unit cell and application of Gauss' integral theorem; the portion of the surface integral over the four sides of the (p,α) solenoids disappears, and the remainder yields a value of $+1$ or -1, depending on whether the three gradients $\nabla p, \nabla\alpha, \nabla\psi$ are oriented in the sense of a right-handed or left-handed screw. (For a related discussion see H. Ertel[2]). Thus, we can interpret

$$\alpha\epsilon_{ijk}\frac{\partial\psi}{\partial x_i}\frac{\partial p}{\partial x_j}\frac{\partial\alpha}{\partial x_k} = \alpha(\nabla p \times \nabla\alpha)\cdot\nabla\psi$$
$$= \pm\frac{\alpha}{V} = N(p,\alpha,\psi) \quad (3.8)$$

as the (positive or negative) number $N(p,\alpha,\psi)$ of (p,α,ψ) unit cells in the specific volume; moreover it follows from (3.7) that

$$\left.\begin{array}{l} N(p,\alpha,\psi) = N(\psi,p,\alpha) = N(\alpha,\psi,p) \\ N(p,\alpha,\psi) = -N(\alpha,p,\psi) = -N(\psi,\alpha,p) \end{array}\right\} . \quad (3.9)$$

Furthermore $N(p,\alpha,\psi) = 0$ for homogeneous fluids $(\nabla\rho = 0)$ while for inhomogeneous fluids $(\nabla\rho \neq 0)$

$$N(p,\alpha,\psi) = 0, \;\text{if} \left\{\begin{array}{l} \psi = \psi(p) \\ \psi = \psi(\alpha) \\ \psi = \psi(p,\alpha) \\ \alpha = \alpha(p) \end{array}\right\} . \quad (3.10)$$

Combining (3.6) and (3.8) we obtain

$$\frac{d}{dt}\left(\alpha\xi_i\frac{\partial\psi}{\partial x_i}\right) - \left(\alpha\xi_i\frac{\partial}{\partial x_i}\right)\frac{d\psi}{dt} = N(p,\alpha,\psi), \quad (3.11)$$

a generalized formulation for hydrodynamical vorticity equations in the commutator relation form

$$(D_1D_2 - D_2D_1)\psi = N(p,\alpha,\psi), \quad (3.12)$$

where the operators

$$D_1 = \frac{d}{dt} = \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) \quad (3.13)$$

and

$$D_2 = \alpha\xi_i\frac{\partial}{\partial x_i} = \alpha\xi_k\frac{\partial}{\partial x_k} = \alpha(\nabla\times\mathbf{v} + 2\mathbf{f})\cdot\nabla \quad (3.14)$$

are in general not commutative.

Multiplying (3.11) by the mass element $\rho d\tau$ ($d\tau =$ volume element), integrating over the volume τ containing the total mass $m = \iiint\rho d\tau$, using $\alpha\rho = 1$, and noting that

$$\iiint \rho\frac{d}{dt}\left(\alpha\xi_i\frac{\partial\psi}{\partial x_i}\right)d\tau = \frac{D}{Dt}\iiint \xi_i\frac{\partial\psi}{\partial x_i}d\tau$$
$$= \frac{D}{Dt}\iiint\frac{\partial(\psi\xi_i)}{\partial x_i}d\tau$$
$$= \frac{D}{Dt}\iint \psi\xi_n d\Omega, \quad (3.15)$$

as well as

$$\iiint \xi_i\frac{\partial}{\partial x_i}\left(\frac{d\psi}{dt}\right)d\tau = \iiint\frac{\partial}{\partial x_i}\left(\xi_i\frac{d\psi}{dt}\right)d\tau$$
$$= \iint \xi_n\frac{d\psi}{dt}d\Omega, \quad (3.16)$$

yields in the integral form ($d\Omega =$ surface element of τ; $n =$ outward normal)

$$\frac{D}{Dt}\iint \xi_n\psi\,d\Omega - \iint \xi_n\frac{d\psi}{dt}d\Omega = N_m(p,\alpha,\psi), \quad (3.17)$$

where $N_m(p,\alpha,\psi) = \iiint\rho N(p,\alpha,\psi)d\tau$ is the number of (p,α,ψ) unit cells contained in the total mass m and $\frac{D}{Dt}$ denotes the individual time derivative of the entire moving mass; in this derivation we note that for any F ($F =$ scalar, vector- or tensor-component) we have

$$\frac{D}{Dt}\iiint \rho F d\tau = \iiint\frac{\partial(\rho F)}{\partial t}d\tau + \iint \rho v_n F d\Omega$$
$$= \iiint \rho\frac{dF}{dt}d\tau, \quad (3.18)$$

since, by means of the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad (3.19)$$

the following transformation is possible:

$$\iiint \frac{\partial (\rho F)}{\partial t} d\tau + \iint \rho v_n F d\Omega$$

$$= \iiint \left\{ \frac{\partial(\rho F)}{\partial t} + \nabla \cdot (\rho \mathbf{v} F) \right\} d\tau$$

$$= \iiint \rho \left\{ \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F \right\} d\tau.$$

Equation (3.18) with $F = \alpha \xi_i (\partial \psi / \partial x_i)$ reverts back to (3.15) when we take into consideration the fact that ξ_i represents a nondivergent vector, i.e.

$$\frac{\partial \xi_i}{\partial x_i} = 0. \qquad (3.20)$$

III. Conclusions

A. If one substitutes for ψ in (3.11) or (3.12) the components x_s $(s = 1, 2, 3)$ of the position vector $\mathbf{r}$ of a fluid particle, one obtains, because $D_1 x_s = v_s$, the vorticity equation (3.2), which for a homogeneous and incompressible fluid, turns into the Helmholtz vorticity equation in the rotating system.

B. If ψ is a conservative property of a fluid particle,

$$D_1 \psi = \frac{d\psi}{dt} = 0, \qquad (3.21)$$

and it then follows from (3.11) or (3.12) that

$$\frac{d}{dt} \left(\alpha \xi_i \frac{\partial \psi}{\partial x_i} \right) = N (p, \alpha, \psi) , \qquad (3.22)$$

which can be stated as follows:

> *For a fluid variable ψ which is individually conserved but has spatial variability, the individual time change of the specific volume multiplied by the scalar product of the absolute vorticity vector with the gradient of ψ is equal to the number of (p, α, ψ) unit cells contained in the specific volume α.*

C. If ψ is a conserved quantity and at the same time has the form

$$\psi = \psi (p, \alpha) , \qquad (3.23)$$

for all fluid particles, then, because

$$\nabla \psi = \frac{\partial \psi}{\partial p} \nabla p + \frac{\partial \psi}{\partial \alpha} \nabla \alpha,$$

(3.10) yields $N (p, \alpha, \psi) = 0$ and (3.22) reduces to

$$\frac{d}{dt} \left(\alpha \xi_i \frac{\partial \psi}{\partial x_i} \right) = 0, \qquad (3.24)$$

which is an analytical statement of the following conservation principle:

> *For a fluid variable ψ which is individually conserved but has a spatial variability such that it depends only on pressure and specific volume, the individual time change of the specific volume multiplied by the scalar product of the absolute vorticity vector with the gradient of ψ vanishes.*

An example of a function ψ which satisfies conditions (3.21) and (3.23) is the entropy of an ideal gas which moves adiabatically; another example is the polytropic temperature[3] of an ideal gas which moves polytropically; the conservation principle is therefore valid for adiabatic or polytropic currents in ideal gases with spatially variable entropy or polytropic temperature (the earth's atmosphere, stellar atmospheres).

D. A homogeneous fluid is defined by $\nabla \alpha = \nabla (1/\rho) = 0$, from which (using (3.8)) it follows that $N (p.\alpha, \psi)$ vanishes for any function ψ; furthermore, $N (p, \alpha, \psi)$ vanishes for any function ψ in the case of barotropy, i.e. $\alpha = \alpha (p)$. Then, there follows from (3.12) the principle:

> *For homogeneous fluids and for inhomogeneous barotropic fluids the operators D_1 and D_2 are always commutative; on the other hand, for inhomogeneous baroclinic fluids, the operators D_1 and D_2 are only commutative for those functions ψ which cause the determinant*
>
> $$\frac{\partial (p, \alpha, \psi)}{\partial (x_1, x_2, x_3)} = (\nabla p \times \nabla \alpha) \cdot \nabla \psi$$
>
> *to vanish.*

References

[1] V. Bjerknes und Mitarbeiter, Physikalische Hydrodynamik, S. 3ff., Berlin 1933.

[2] H. Ertel, Meteorol. Zeitschr., **59**, 1942.

[3] R. Emden, Enzykl. Math. Wiss., VI, 2, 24, Leipzig und Berlin 1926.

4 ERTEL (1942d): On the relationship between the new hydrodynamic vorticity theorem and Bjerknes' circulation theorem

Summary

It is proven that Bjerknes' circulation theorem is a special case of the new hydrodynamical vorticity theorem.

1. The new vorticity theorem

Recently I derived a very general hydrodynamic vorticity theorem [1] [2]. In its differential form it reads as follows:

$$\frac{d}{dt}(\sigma\mathfrak{W}\cdot\nabla\psi) - (\sigma\mathfrak{W}\cdot\nabla)\frac{d\psi}{dt} = N(p,\sigma,\psi).$$
(4.1)

It has the form of a commutation relation

$$(D_1 D_2 - D_2 D_1)\psi = N(p,\sigma,\psi)$$
(4.2)

with the generally noncommutative operators

$$D_1 = \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$
(4.3)

and

$$D_2 = (\sigma\mathfrak{W}\cdot\nabla) = \sigma(\operatorname{rot}\mathbf{v} + 2\mathbf{f})\cdot\nabla,$$
(4.4)

where the symbols used in (4.1) through (4.4) are defined as follows: σ = specific volume = $1/\rho$ (ρ = density); p = pressure; $\mathbf{v}$ = velocity vector relative to a coordinate system, its rotation described through the constant rotation vector $\mathbf{f}$; $\mathfrak{W}$ = $\operatorname{rot}\mathbf{v} + 2\mathbf{f}$ = (absolute) vorticity; ψ = hydrodynamic field function (scalar, vector, or tensor component); $N(p,\sigma,\psi)$ = $\sigma[\nabla p,\nabla\sigma]\cdot\nabla\psi$ = number of (p,σ,ψ)-cells contained within a specific volume σ.

For the derivation of (4.1), it has been assumed that the fluid is ideal and that external forces acting on the fluid possess a potential. The sign of $N(p,\sigma,\psi)$ is positive or negative depending on whether the generally nonorthogonal vectors ∇p, $\nabla\sigma$ and $\nabla\psi$ are oriented in a right- or left-handed system.

One can obtain equations for each component of vorticity by specifying in (4.1) the function ψ to have a purely geometric interpretation. Namely, one successively sets $\psi = x$, $\psi = y$, and $\psi = z$ in such a way that from (4.1) the following vorticity equations result:

$$\left.\begin{aligned}
\frac{d}{dt}(\sigma\mathfrak{W}_x) - (\sigma\mathfrak{W}\cdot\nabla)v_x &= \sigma[\nabla p,\nabla\sigma]_x, \\
\frac{d}{dt}(\sigma\mathfrak{W}_y) - (\sigma\mathfrak{W}\cdot\nabla)v_y &= \sigma[\nabla p,\nabla\sigma]_y, \\
\frac{d}{dt}(\sigma\mathfrak{W}_z) - (\sigma\mathfrak{W}\cdot\nabla)v_z &= \sigma[\nabla p,\nabla\sigma]_z,
\end{aligned}\right\}$$
(4.5)

where, for instance,

$$[\nabla p,\nabla\sigma]_x = \frac{\partial p}{\partial y}\frac{\partial\sigma}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial\sigma}{\partial y}$$
(4.6)

is the number of the isobar-isostere unit solenoids per unit surface area normal to the $+x$ direction. For irrotational coordinates ($\mathbf{f} \equiv 0$) and barotropy ($p = p(\sigma)$), (4.5) yields the vorticity equations of E. J. NANSON (1874); furthermore, adding incompressibility, one obtains the vorticity equations of HELMHOLTZ (1858) [LAGRANGE (1781), CAUCHY (1827), STOKES (1848)].

The integral form of the new vorticity theorem reads (Ertel, l.c.):

$$\frac{d}{dt}\iint\mathfrak{W}_n\psi d\Omega - \iint\mathfrak{W}_n\frac{d\psi}{dt}d\Omega = N_m(p,\sigma,\psi).$$
(4.7)

Here,

$$\begin{aligned}
N_m(p,\sigma,\psi) &= \iiint\rho N(p,\sigma,\psi)d\tau \\
&= \iiint[\nabla p,\nabla\sigma]\cdot\nabla\psi d\tau
\end{aligned}$$
(4.8)

is the number of (p,σ,ψ)-unit cells, $m = \iiint\rho d\tau$ is the mass (with volume τ contained in the surface Ω), and n in (4.7) represents the outward normal. Because

$$[\nabla p,\nabla\sigma]\cdot\nabla\psi = \nabla\cdot(\psi[\nabla p,\nabla\sigma]),$$

(4.8) can be transformed through Gauss' integral theorem into

$$N_m(p,\sigma,\psi) = \iint\psi[\nabla p,\nabla\sigma]_n d\Omega.$$
(4.9)

Then, the integral form (4.7) of the new vorticity theorem changes into

$$\begin{aligned}
\frac{d}{dt}\iint\mathfrak{W}_n\psi d\Omega &- \iint\mathfrak{W}_n\frac{d\psi}{dt}d\Omega \\
&= \iint\psi[\nabla p,\nabla\sigma]_n d\Omega.
\end{aligned}$$
(4.10)

One assumes in (4.7) and (4.10) that the functions ψ, $d\psi/dt$, $\mathfrak{W}$ and $[\nabla p,\nabla\sigma]$ are defined on the surface Ω and in the volume τ the first derivatives are continuous

with respect to the spatial coordinates; the closed surface Ω can itself consist of finitely many pieces of continuous tangential planes.

2. The new vorticity theorem with a discontinuous ψ-function

Consider discontinuous ψ and $d\psi/dt$ on the surface Σ. We denote the jump in ψ as

$$\{\psi\} = \psi_{+0} - \psi_{-0} \tag{4.11}$$

and the jump in $d\psi/dt$ as

$$\left\{\frac{d\psi}{dt}\right\} = \left(\frac{d\psi}{dt}\right)_{+0} - \left(\frac{d\psi}{dt}\right)_{-0} \tag{4.12}$$

on the surface Σ (the indices $+0$ and -0 specify the function values directly above and directly below the surface Σ, respectively). Then, we construct a cylinder with face F parallel to Σ and the infinitesmal height h. Applying the integral form (4.10) of the new vorticity theorem to the region $F \cdot h$ and taking the limit as $h \to 0$, the integral over the lateral surface vanishes and we obtain the following:

$$\frac{d}{dt} \iint \mathfrak{W}_n \psi_{-0} dF - \iint \mathfrak{W}_n \left(\frac{d\psi}{dt}\right)_{+0} dF$$
$$+ \frac{d}{dt} \iint \mathfrak{W}_{-n} \psi_{-0} dF - \iint \mathfrak{W}_{-n} \left(\frac{d\psi}{dt}\right)_{-0} dF$$
$$= \iint \psi_{+0} [\nabla p, \nabla \sigma]_n dF$$
$$+ \iint \psi_{-0} [\nabla p, \nabla \sigma]_{-n} dF \tag{4.13}$$

By the continuity of $\mathfrak{W}$ and $[\nabla p, \nabla \sigma]$ on Σ (and thus on the subarea F of Σ)

$$\mathfrak{W}_{-n} = -\mathfrak{W}_n$$

and

$$[\nabla p, \nabla \sigma]_{-n} = [\nabla p, \nabla \sigma]_n.$$

Then, in consideration of (4.11) and (4.12) we can write

$$\boxed{\begin{array}{c} \dfrac{d}{dt} \iint \mathfrak{W}_n \{\psi\} dF - \iint \mathfrak{W}_n \left\{\dfrac{d\psi}{dt}\right\} dF \\ = \iint \{\psi\} [\nabla p, \nabla \sigma]_n dF. \end{array}} \tag{4.14}$$

Equation (4.14) represents the integral form of the new vorticity theorem for a field function ψ and discontinuous derivative $d\psi/dt$ on the surface F.

If, on the surface F, only the field function ψ becomes discontinuous;

$$\{\psi\} = \psi_{+0} - \psi_{-0} \neq 0,$$

while $d\psi/dt$, however, remains continuous:

$$\left\{\frac{d\psi}{dt}\right\} = \left(\frac{d\psi}{dt}\right)_{+0} - \left(\frac{d\psi}{dt}\right)_{-0} = 0,$$

then (4.14) reduces to

$$\boxed{\frac{d}{dt} \iint \{\psi\} \mathfrak{W}_n dF = \iint \{\psi\} [\nabla p, \nabla \sigma]_n dF.} \tag{4.15}$$

By properly specifying the function ψ, Bjerknes' circulation theorem results from equation (4.15).

3. The circulation theorem of V. Bjerknes as a special case of the new vorticity theorem

The surface Σ, which contains the surface F, divides the fluid into an "upper fluid half-space" R_1 and a "lower fluid half-space" R_2. The surface Σ, and with it F, moves with the current, such that particles which are in R_1 always remain in R_1 and particles in R_2 always remain in R_2, while the common boundary Σ (with the surface F) of the two fluid spaces R_1 and R_2 deforms itself according to the flow. We now imagine that each fluid particle in R_1 is assigned a same constant number Z_1; thus, this number identifies each individual fluid element belonging to the fluid half-space R_1. Similarly, we imagine all fluid particles from R_2 possessing the same constant number Z_2. The numbers Z_1 and Z_2 do not characterize the fluid particles individually, because, for instance, all fluid particles in R_1 are assigned the same number Z_1; rather, the numbers Z_1 and Z_2 are group distinguisher indices that decide which of the two applicable groups of fluid particles each particle belongs to. However, each fluid element maintains the assigned number of Z_1 or Z_2 individually:

$$\frac{d}{dt} Z_1 = \frac{d}{dt} Z_2 = 0.$$

For example, a particle initially assigned a number Z_1 and belonging to the fluid half-space R_1 cannot belong to R_2 later and not have the assigned number Z_2 with it, since the particle would have to cross the surface Σ. This is impossible since the same number is always associated with the same particle, if Σ, as adopted, moves along with the flow. However, the number (either Z_1 or Z_2) changes discontinuously at the surfaces Σ and F as $|Z_1 - Z_2| \neq 0$. The restriction of a fluid element to either the fluid half-space R_1 or to the fluid half-space R_2 with these two characteristic numbers Z_1
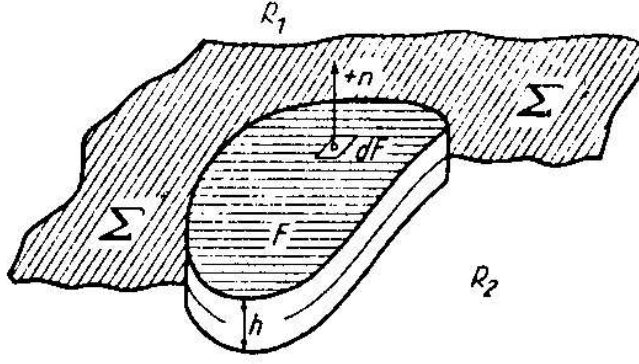
13

Fig. 4.1

and Z_2 thus forms a discontinuous ψ-field with the characteristics

$$\psi = Z_1, \quad \frac{d\psi}{dt} = 0 \quad \text{(in } R_1\text{)},$$

$$\psi = Z_2, \quad \frac{d\psi}{dt} = 0 \quad \text{(in } R_2\text{)},$$

as well as

$$\begin{aligned} \{\psi\} &= \psi_{+0} - \psi_{-0} \\ &= Z_1 - Z_2 \neq 0 \quad \text{(on } \Sigma \text{ and/or } F\text{)} \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \left\{\frac{d\psi}{dt}\right\} &= \left(\frac{d\psi}{dt}\right)_{+0} - \left(\frac{d\psi}{dt}\right)_{-0} \\ &= 0, \quad \text{(on } \Sigma \text{ and/or } F\text{)} \end{aligned} \tag{4.17}$$

where for this ψ-function the new vorticity theorem of the form (4.15) applies. Since $\{\psi\} = Z_1 - Z_2$ is a constant jump value on Σ and/or F, the equation (4.15) immediately results in

$$\begin{aligned} (Z_1 - Z_2)\frac{d}{dt}\iint \mathfrak{W}_n dF \\ = (Z_1 - Z_2)\iint [\nabla p, \nabla \sigma]_n dF, \end{aligned}$$

or through division by $Z_1 - Z_2 \neq 0$,

$$\boxed{\frac{d}{dt}\iint \mathfrak{W}_n dF = \iint [\nabla p, \nabla \sigma]_n dF,} \tag{4.18}$$

which is the equation that represents Bjerknes' circulation theorem.

We note that

$$\iint [\nabla p, \nabla \sigma]_n dF = N(p, \sigma) \tag{4.19}$$

is the quantity of isobar-isostere unit solenoids in F and

$$\begin{aligned} \iint \mathfrak{W}_n dF &= \iint (\mathrm{rot}_n \mathbf{v} + 2\mathbf{f}_n) dF \\ &= \oint v_s ds + 2\omega S \end{aligned} \tag{4.20}$$

is the absolute circulation along the boundary of F. Then, through Stokes' theorem,

$$\iint \mathrm{rot}_n \mathbf{v}\, dF = \oint v_s\, ds = C$$

is the circulation C relative to the Earth (rotating system). Furthermore,

$$\begin{aligned} 2\iint \mathbf{f}_n dF &= 2\iint \omega \cos(n, \mathbf{f}) dF \\ &= 2\omega S \end{aligned}$$

is twice the angular velocity $\omega = |\mathbf{f}|$ multiplied by the projection $S = \iint \cos(n, \mathbf{f}) dF$ of the area F onto any plane (e.g., equatorial plane) perpendicular to the rotation axis, and thus (4.18) is equivalent with the usual version of the Bjerknes' circulation theorem:

$$\frac{dC}{dt} + 2\omega \frac{dS}{dt} = N(p, \sigma). \tag{4.21}$$

Following (4.15) one obtains Lord Kelvin's circulation theorem for irrotational coordinates ($\mathbf{f} = 0$) and a barotropic fluid ($p = p(\sigma)$) [W. THOMSON (1869)]:

$$\frac{d}{dt}\oint v_s ds = 0. \tag{4.22}$$

In summary, from the new hydrodynamic vorticity theorem in its differential and integral forms, all well-known vorticity and circulation equations, respectively, can be derived as special cases. In addition, the new vorticity theorem results in still unknown special cases, for example, a conservation law valid for polytropic flows [3] (ψ = polytropic temperature), for which F. MORÀN [4] has furnished a very appealing proof.

References

1] H. Ertel, Met. Z. 1942.

2] H. Ertel, Physik. Z. 1942.

3] H. Ertel, Die Naturwissenschaften 1942.

4] F. Moràn, Revista de Geofisica 1942.

5   ERTEL (1952): On the physical significance of functions arising in the Clebsch transformation of the hydrodynamical equations

I. Geometry and kinematics of a stationary barotropic fluid

In the Clebsch transformation of the hydrodynamic equations [Ref. (1, 2, 3)] the wind velocity vector $\mathbf{v}$ is represented by the superposition of two vectors [Ref. (4)] involving the three scalar functions φ, λ, μ:

$$\mathbf{v} = \nabla\varphi + \lambda\nabla\mu, \qquad (5.1)$$

from which it follows that the functions λ and μ are related to the vorticity vector $\boldsymbol{\xi} = \nabla \times \mathbf{v}$ in accordance with

$$\boldsymbol{\xi} = \nabla\lambda \times \nabla\mu. \qquad (5.2)$$

Thus, at each point in the flow the direction of the vorticity vector is along the intersection of the two surfaces $\lambda = \text{const}$ and $\mu = \text{const}$.

At the same time, this intersection curve itself crosses the energy surface

$$H = \frac{v^2}{2} + \Phi + \int \frac{dp}{\rho} = \text{const}, \qquad (5.3)$$

which contains the regarded point if, as is always assumed in the following, we consider *the stationary flow of an ideal barotropic fluid*. In the expression (5.3) for the total energy H per unit mass, we define $v = |\mathbf{v}|$ and Φ as the potential of external forces. For all fluid particles, the presupposed barotropy is ensured by the existence of a single piezotropic relation $\rho = \rho(p)$ ($\rho =$ density, $p =$ pressure). The sum $\Phi + \int dp/\rho$ represents the total potential energy per unit mass.

In Eulerian form fluid movement is described by the equation

$$\frac{D\mathbf{v}}{Dt} = -\nabla\left(\Phi + \int \frac{dp}{\rho}\right) \qquad (5.4)$$

with the material derivative operator

$$\mathbf{v} \cdot \nabla = \frac{D}{Dt}, \qquad (5.5)$$

or by the equation

$$\frac{D\mathbf{v}}{Dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{v^2}{2}\right) - \mathbf{v} \times \nabla \times \mathbf{v}, \quad (5.6)$$

which is equivalent to (5.4) because

$$\mathbf{v} \times \boldsymbol{\xi} = \nabla H. \qquad (5.7)$$

Scalar multiplication of (5.7) with (5.2) results in (x, y, z are orthogonal cartesian coordinates):

$$\boldsymbol{\xi} \cdot \nabla H = \nabla H \cdot (\nabla\lambda \times \nabla\mu) = \frac{\partial(H, \lambda, \mu)}{\partial(x, y, z)} = 0, \quad (5.8)$$

an equation which shows the geometric collapse of the surface system $\lambda = \text{const}$, $\mu = \text{const}$, $H = \text{const}$, and which leads analytically to the existence of a functional relationship

$$H = H(\lambda, \mu). \qquad (5.9)$$

Scalar multiplication of the momentum equation (5.7) with the wind velocity vector $\mathbf{v}$ yields

$$\mathbf{v} \cdot \nabla H = \frac{DH}{Dt} = 0, \qquad (5.10)$$

which represents the conservation law for the total energy and which expresses in a geometric sense a coincidence of the streamlines with the energy surface. Vorticity lines (with tangents $\boldsymbol{\xi}$) and streamlines (with tangents $\mathbf{v}$) form a grid on each energy surface, whereby the angle of intersection (the angle between $\mathbf{v}$ and $\boldsymbol{\xi}$) is determined by the equation

$$|\mathbf{v}| \cdot |\boldsymbol{\xi}| \cdot \sin(\mathbf{v}, \boldsymbol{\xi}) = \frac{\partial H}{\partial n}, \qquad (5.11)$$

which follows from (5.7), with n denoting the positive normal (pointing to larger energy values) to the energy surface. If δn denotes the distance between two neighboring energy surfaces of constant energy difference ($\frac{\partial H}{\partial n}\delta n = \text{const}$), the equation

$$\delta n \cdot |\mathbf{v}| \cdot |\boldsymbol{\xi}| \cdot \sin(\mathbf{v}, \boldsymbol{\xi}) = \text{const}, \qquad (5.12)$$

which follows from (5.11), can characterize the flow in terms of the streamline/vorticity grid on the energy surface [Ref. (5, 6, 7)]. By the introduction of unit volumes, formed from energy surfaces and the streamline/vorticity grid, these views can be formulated in an even simpler way [Ref. (8)].

The substitution of (5.2) and (5.9) into (5.7) results in the relation

$$\mathbf{v} \times (\nabla\lambda \times \nabla\mu) = \frac{\partial H}{\partial\lambda}\nabla\lambda + \frac{\partial H}{\partial\mu}\nabla\mu, \qquad (5.13)$$

or equivalently,

$$(\mathbf{v}\cdot\nabla\mu)\nabla\lambda - (\mathbf{v}\cdot\nabla\lambda)\nabla\mu = \frac{\partial H}{\partial\lambda}\nabla\lambda + \frac{\partial H}{\partial\mu}\nabla\mu, \quad (5.14)$$

15

and the existence of Hamilton's canonical equations

$$\frac{D\mu}{Dt} = \frac{\partial H}{\partial \lambda}, \qquad \frac{D\lambda}{Dt} = -\frac{\partial H}{\partial \mu} \qquad (5.15)$$

follows [Ref. (9)] for the functions (λ, μ). The above connection of the functions λ, μ with the energy function H now permits the conclusion that the functions λ, μ must possess a physical interpretation that goes beyond their geometric-kinematic meaning (as determinants of the vorticity lines); this conclusion can also be expanded to the function φ in equation (5.1).

II. Transition to the Lagrangian form of hydrodynamics

For the retrieval of the physical meaning of the functions φ, λ, μ it is advisable to use a transformation of the Eulerian equations (5.4) to the Lagrangian equations

$$\frac{\partial^2 x_j}{\partial t^2}\frac{\partial x_j}{\partial a_i} = -\frac{\partial}{\partial a_i}\left(\Phi + \int \frac{dp}{\rho}\right) \qquad (i, j = 1, 2, 3).$$
$$(5.16)$$

Here,

$$x_j = x_j(a_1, a_2, a_3, t) \qquad (j = 1, 2, 3) \qquad (5.17)$$

are the cartesian position coordinates at time t and a_1, a_2, a_3 (labeling coordinates) are the position coordinates at time $t = 0$, which in a broader sense characterize the motion of individual fluid particles. In (5.16) the Einstein summation convention has been used, so that if an index appears twice, it is summed from 1 to 3. Together with the continuity equation

$$\rho\,\frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} = \rho_0 = \rho(a_1, a_2, a_3, 0) \qquad (5.18)$$

and the piezotropic relation

$$\rho = \rho(p), \qquad (5.19)$$

equations (5.16, 5.18, 5.19) form a system of five equations for the three functions (5.17), the pressure p and the density ρ.

Now let
$$F(\overset{*}{x}_1, \overset{*}{x}_2, \overset{*}{x}_3) = 0 \qquad (5.20)$$

be the equation of a surface lying in the space, related to the same orthogonal cartesian system x_i $(i = 1, 2, 3)$, which serves for the determination of the position coordinates (5.17); the variables with attached stars denote those coordinates which satisfy the surface equation (5.20). The surface may possess a continuously variable tangent plane and be oriented in such a manner that it is punctured by the streamlines, but the exact alignment of the surface relative to the streamline system is as yet unspecified.

The position coordinates (5.17) of a particle with initial coordinates a_1, a_2, a_3 must, at the time ϑ when this particle lies on the surface $F(\overset{*}{x}_1, \overset{*}{x}_2, \overset{*}{x}_3) = 0$, agree with the coordinates $\overset{*}{x}_1, \overset{*}{x}_2, \overset{*}{x}_3$, so that:

$$\overset{*}{x}_j = x_j(a_1, a_2, a_3, \vartheta) \qquad (j = 1, 2, 3) \qquad (5.21)$$

and the equation of the surface becomes

$$\begin{aligned} F(x_1(a_1, a_2, a_3, \vartheta), \\ x_2(a_1, a_2, a_3, \vartheta), \\ x_3(a_1, a_2, a_3, \vartheta)) = 0. \end{aligned} \qquad (5.22)$$

The solution of this equation for ϑ yields:

$$\vartheta = \vartheta(a_1, a_2, a_3), \qquad (5.23)$$

which indicates the time at which an individually labeled particle a_1, a_2, a_3 lies in the given surface $F = 0$.

For the following argument it is useful to define the wind components

$$v_j = \frac{\partial x_j}{\partial t} \qquad (j = 1, 2, 3), \qquad (5.24)$$

the kinetic energy (per unit mass)

$$\frac{1}{2}\left(\frac{\partial x_j}{\partial t}\right)^2 = \frac{1}{2}v_j v_j = \frac{1}{2}v^2, \qquad (5.25)$$

and the Lagrangian function

$$L = \frac{1}{2}v^2 - \left(\Phi + \int \frac{dp}{\rho}\right), \qquad (5.26)$$

and thereby rewrite the Lagrangian equations (5.16) in the following form:

$$\frac{\partial}{\partial t}\left(v_j \frac{\partial x_j}{\partial a_i}\right) = \frac{\partial L}{\partial a_i}. \qquad (5.27)$$

III. The physical meaning of the functions φ, λ, μ

We integrate equations (5.27) from $t = \vartheta(a_1, a_2, a_3)$ to time $t > \vartheta$ and obtain

$$v_j \frac{\partial x_j}{\partial a_i} = (v_j)_\vartheta \left(\frac{\partial x_j}{\partial a_i}\right)_\vartheta + \int_\vartheta^t \frac{\partial L}{\partial a_i}\,dt. \qquad (5.28)$$

Through differentiation with respect to a_i it follows from (5.21) that

$$\frac{\partial \overset{*}{x}_j}{\partial a_i} = \left(\frac{\partial x_j}{\partial a_i}\right) + (v_j)_\vartheta \frac{\partial \vartheta}{\partial a_i}, \qquad (5.29)$$

16

and therefore (5.28) yields:

$$v_j \frac{\partial x_j}{\partial a_i} = (v_j)_\vartheta \frac{\partial \overset{*}{x}_j}{\partial a_i}$$
$$+ \frac{\partial}{\partial a_i} \int_\vartheta^t L \, dt + \left\{ (L)_\vartheta - (v_j)_\vartheta^2 \right\} \frac{\partial \vartheta}{\partial a_i}, \quad (5.30)$$

where we have used the formula

$$\frac{\partial}{\partial a_i} \int_\vartheta^t L \, dt = \int_\vartheta^t \frac{\partial L}{\partial a_i} \, dt - (L)_\vartheta \frac{\partial \vartheta}{\partial a_i}. \quad (5.31)$$

Now, in (5.30)

$$\int_\vartheta^t L \, dt = W \quad (5.32)$$

represents the Hamiltonian action function [Ref. (10, 11, 12)], and furthermore (with $v_j^2 = v_j v_j = v^2$)

$$(L)_\vartheta - (v_j)_\vartheta^2 = - \left\{ \frac{v^2}{2} + \left(\Phi + \int \frac{dp}{\rho} \right) \right\} \quad (5.33)$$
$$= -H_\vartheta = -H$$

is the negative of the total energy (per unit mass) of the observed particle at time ϑ, which equals ($H_\vartheta = H$), because of the conservation law (5.10), the negative total energy $-H$ for $t \geq \vartheta$, so that (5.30) takes the form

$$v_j \frac{\partial x_j}{\partial a_i} = (v_j)_\vartheta \frac{\partial \overset{*}{x}_j}{\partial a_i} + \frac{\partial W}{\partial a_i} - H \frac{\partial \vartheta}{\partial a_i}. \quad (5.34)$$

Equation (5.34) is the result of the integration of the Lagrangian momentum equations (5.16), or equivalently (5.27), over the time interval of the movement of a particle along a certain streamline segment. We now consider a closely-adjoining streamline and on this streamline a particle that at time $t = 0$ has the associated labeling-coordinate $a_i + \delta a_i$ ($i = 1, 2, 3$). With

$$\frac{\partial x_j}{\partial a_i} \delta a_i = \delta x_j, \qquad \frac{\partial \overset{*}{x}_j}{\partial a_i} \delta a_i = \delta \overset{*}{x}_j \quad (5.35)$$

and

$$\frac{\partial W}{\partial a_i} \delta a_i = \delta W, \qquad \frac{\partial \vartheta}{\partial a_i} \delta a_i = \delta \vartheta, \quad (5.36)$$

it follows from (5.34) that

$$v_j \delta x_j = (v_j)_\vartheta \, \delta \overset{*}{x}_j + \delta W - H \delta \vartheta, \quad (5.37)$$

from which we can write

$$v_j \delta x_j = (v_j)_\vartheta \, \delta \overset{*}{x}_j + \delta W + H \delta(t - \vartheta), \quad (5.38)$$

since the end time t is not varied. Now $\delta \overset{*}{x}_j$ ($j = 1, 2, 3$) denotes the infinitesimal vector that extents from

the intersection point $P(\overset{*}{x}_1, \overset{*}{x}_2, \overset{*}{x}_3)$ of the first streamline with the surface $F = 0$ to the intersection point $P(\overset{*}{x}_1 + \delta \overset{*}{x}_1, \overset{*}{x}_2 + \delta \overset{*}{x}_2, \overset{*}{x}_3 + \delta \overset{*}{x}_3)$ of the second streamline with the surface $F = 0$. We now specify the situation, which was not yet defined in section II, concerning the orientation of the surface $F = 0$ relative to the streamline system, by demanding *that the surface $F = 0$ should cut the streamline system orthogonally.* Then

$$(v_j)_\vartheta \, \delta \overset{*}{x}_j = 0, \quad (5.39)$$

and equation (5.38) reduces to

$$v_j \, \delta x_j = \delta W + H \delta \Theta, \quad (5.40)$$

where

$$\Theta = t - \vartheta(a_1, a_2, a_3) \quad (5.41)$$

denotes the running time of the particle (a_1, a_2, a_3) from the time $\vartheta(a_1, a_2, a_3)$ of its intersection with the orthogonal surface $F = 0$ to the time t.

We now go back to the Eulerian form of hydrodynamics and select the cartesian coordinates x_j ($j = 1, 2, 3$) as arguments, so that from (5.40) we obtain

$$v_j \, \delta x_j = \left(\frac{\partial W}{\partial x_j} + H \frac{\partial \Theta}{\partial x_j} \right) \delta x_j, \quad (5.42)$$

and then because of the free choice of δa_i or δx_j it follows that

$$v_j = \frac{\partial W}{\partial x_j} + H \frac{\partial \Theta}{\partial x_j}, \quad (5.43)$$

or in vector form

$$\boxed{\mathbf{v} = \nabla W + H \nabla \Theta.} \quad (5.44)$$

Hence, the task is solved, since a comparison with (5.1) results in:

$$\boxed{\begin{aligned} \varphi = W &= \int_\vartheta^t L \, dt \\ &= \int_\vartheta^t \left\{ \frac{v^2}{2} - \left(\Phi + \int \frac{dp}{\rho} \right) \right\} dt \\ &= \text{Hamilton's action function,} \end{aligned}} \quad (5.45)$$

$$\boxed{\begin{aligned} \lambda = H &= \frac{v^2}{2} + \left(\Phi + \int \frac{dp}{\rho} \right) \\ &= \text{total energy (per unit mass),} \end{aligned}} \quad (5.46)$$

$$\boxed{\begin{aligned} \mu = \Theta &= t - \vartheta(a_1, a_2, a_3) \\ &= \text{running time,} \end{aligned}} \quad (5.47)$$

with the latter measured from the time $\vartheta(a_1, a_2, a_3)$ of the intersection of the particle (a_1, a_2, a_3) with the surface that cuts the streamline system orthogonally. Which of the infinitely many orthogonal surfaces is selected as the "zero-surface" for measuring the running time Θ is unimportant for the computation of the stationary velocity field from equation (5.44), if in the area between two such orthogonal surfaces the hydrodynamic momentum equations (5.4), or equivalently (5.7), apply. It should be noted that the Hamiltonian function W in (5.44) is to be formed for the running time interval $t - \vartheta = \Theta$.

I will discuss elsewhere the modified interpretation of φ, λ, μ of the representation (5.44) which occurs in the case of non-stationary flows.

Scalar multiplication of equation (5.44) with the velocity vector $\mathbf{v}$ results in the identity $v^2 = v^2$, because, first of all, in consideration of (5.5):

$$v^2 = \frac{DW}{Dt} + H\frac{D\Theta}{Dt} \qquad (5.48)$$

and furthermore, according to (5.32) and (5.41):

$$\frac{DW}{Dt} = L = \frac{v^2}{2} - \left(\Phi + \int \frac{dp}{\rho}\right), \qquad \frac{D\Theta}{Dt} = 1, \qquad (5.49)$$

and

$$\frac{DW}{Dt} + H\frac{D\Theta}{Dt} = L + H = v^2. \qquad (5.50)$$

The Hamiltonian canonical equations (5.15) are identically satisfied, because, with (5.46, 5.47):

$$\frac{D\mu}{Dt} = 1, \qquad \frac{\partial H}{\partial \lambda} = 1, \qquad (5.51)$$

and from (5.10, 5.46, 5.47):

$$\frac{D\lambda}{Dt} = 0, \qquad \frac{\partial H}{\partial \mu} = 0. \qquad (5.52)$$

IV. Introduction of the "reduced action function"

We now introduce in (5.44) the "reduced action function" defined by [Ref. (13)]

$$S = W + H\Theta = \int_\vartheta^t (L + H)\, dt$$
$$= \int_\vartheta^t v^2\, dt = \int_B^{B+\Delta B} v\, ds, \qquad (5.53)$$

where $ds = v\, dt$ denotes a line element of the trajectory (also a streamline in the case of stationary motion) and ΔB is the length of this trajectory from the intersection point with the orthogonal surface $F = 0$ up to the

point (x_1, x_2, x_3). Thus, from (5.44), one now obtains the representation

$$\boxed{\mathbf{v} = \nabla S - \Theta \nabla H,} \qquad (5.54)$$

or written in more detail:

$$\mathbf{v} = \nabla\left(\int_B^{B+\Delta B} v\, ds\right) - \left(\int_B^{B+\Delta B} \frac{ds}{v}\right)\nabla H, \qquad (5.55)$$

where the running time (the elapsed time) is given by

$$\Theta = \int_B^{B+\Delta B} \frac{ds}{v} \qquad (5.56)$$

($B = $ the length of the trajectory from a_i to $\overset{*}{x}_i$; $i = 1, 2, 3$.)

V. Application to asynchronous-periodical flows

In a stationary asynchronous-periodical flow field, the fluid particles pass through closed streamlines (trajectories, particle tracks) with circulation times τ or frequencies $\nu = 1/\tau$ that differ for trajectories on different energy surfaces. Now the velocity vector $\mathbf{v}$ for a particle on a closed orbit C of the length $K = \int_C ds$ changes periodically after the circulation time, so from (5.54, 5.55) we have ($B_1 = B + \Delta B$):

$$\mathbf{v} = \nabla\left(\int_B^{B_1} v\, ds\right) - \Theta\nabla H$$
$$= \nabla\left(\int_B^{B_1+K} v\, ds\right) - (\Theta + \tau)\nabla H, \qquad (5.57)$$

and also

$$0 = \nabla\left(\int_{B_1}^{B_1+K} v\, ds\right) - \tau\nabla H. \qquad (5.58)$$

Because

$$\int_{B_1}^{B_1+K} v\, ds = \oint v\, ds = \Gamma \qquad (5.59)$$

is the circulation along the closed path, one finds, after scalar multiplication of equation (5.58) with a differential vector $d\mathbf{r} = (dx_1, dx_2, dx_3)$, that the changes dH and $d\Gamma$ from an arbitrary point on an orbit lying in the energy surface H with the circulation Γ to a point on the energy surface $H + dH = H + d\mathbf{r} \cdot \nabla H$ with the circulation $\Gamma + d\Gamma = \Gamma + d\mathbf{r} \cdot \nabla\Gamma$ leads, are related by

$$\tau = \frac{d\Gamma}{dH}, \qquad (5.60)$$

which represents the hydrodynamic generalization of a result from the particle dynamics of single periodic motions in relationship with the Hamilton-Jacobi theory [Ref. (14)].

VI. Relation to the hydrodynamical conservation principle of H. Ertel and C.-G. Rossby

One can define

$$\hat{W} = \int_0^t L \, dt \qquad (5.61)$$

as the action function with lower integration limit $t = 0$. Then, even for non-stationary flows, the Ertel-Rossby [Ref. (15, 16)] conservation law

$$\frac{d}{dt}\left\{ \sigma \boldsymbol{\xi} \cdot (\mathbf{v} - \nabla \hat{W}) \right\} = 0, \qquad (5.62)$$

holds ($\sigma = 1/\rho$ = specific volume) with the Eulerian material derivative operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \qquad (5.63)$$

For stationary flow fields, d/dt must be replaced with the operator $\mathbf{v} \cdot \nabla = D/Dt$, so that

$$\frac{D}{Dt}\left\{ \sigma \boldsymbol{\xi} \cdot (\mathbf{v} - \nabla \hat{W}) \right\} = 0, \qquad (5.64)$$

a result which can be easily deduced from (5.44). To see this, we form, out of (5.44) and in accordance with (5.41), the corresponding equations

$$\mathbf{v} = \nabla W - H \nabla \vartheta, \qquad (5.65)$$

and

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = -\nabla H \times \nabla \vartheta. \qquad (5.66)$$

Furthermore, through the introduction of

$$\hat{W} = \int_0^t L \, dt = \int_0^\vartheta L \, dt + \int_\vartheta^t L \, dt$$
$$= \overset{*}{W} + W, \qquad (5.67)$$

where, in accordance with (5.23),

$$\overset{*}{W} = \int_0^\vartheta L \, dt \qquad (5.68)$$

is independent of the time $t > \vartheta$, we can rewrite (5.65) as

$$\mathbf{v} - \nabla \hat{W} = -\nabla \overset{*}{W} - H \nabla \vartheta. \qquad (5.69)$$

Then scalar multiplication of (5.66) with (5.69) yields first of all:

$$\boldsymbol{\xi} \cdot (\mathbf{v} - \nabla \hat{W}) = \nabla \overset{*}{W} \cdot (\nabla H \times \nabla \vartheta)$$
$$= \frac{\partial(\overset{*}{W}, H, \vartheta)}{\partial(x_1, x_2, x_3)}, \qquad (5.70)$$

and further through multiplication with the continuity equation (5.18), with the aid of the multiplication rule for the functional determinants:

$$\rho_0 \, \boldsymbol{\xi} \cdot (\mathbf{v} - \nabla \hat{W}) = \rho \, \frac{\partial(\overset{*}{W}, H, \vartheta)}{\partial(a_1, a_2, a_3)}. \qquad (5.71)$$

Through introduction of the specific volumes (σ, σ_0), one obtains the formula

$$\sigma \boldsymbol{\xi} \cdot (\mathbf{v} - \nabla \hat{W}) = \sigma_0 \, \frac{\partial(\overset{*}{W}, H, \vartheta)}{\partial(a_1, a_2, a_3)}, \qquad (5.72)$$

in which the functions appearing on the right side are independent of time on each marked particle (a_1, a_2, a_3), so that the use of the D/Dt-operator yields the form (5.64) of the conservation relation.

VII. Vortex lines

From (5.44) it follows that

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = \nabla H \times \nabla \Theta, \qquad (5.73)$$

i.e., the vorticity lines are curves which are formed by the intersections of the running time surfaces $\Theta = \text{const}$ with the energy surfaces $H = \text{const}$.

VIII. Summary

The physical meaning of the three scalar functions (φ, λ, μ) appearing in the Clebsch-transformation of the hydrodynamic equations is determined by means of the Lagrangian for the case of stationary flow in a barotropic fluid. The flow is determined by the functions λ = total energy, μ = elapsed time after the fluid particle penetrates a surface orthogonal to the streamline system, and φ = the Hamiltonian action integral, with the integration extended over the running time. The determination of the physical meaning of these functions enables in a more simple manner the derivation of a theorem concerning asynchronous-periodical flows as well as the proof of the Ertel-Rossby conservation relation of hydrodynamics.

The physical interpretation given here for the three scalar functions appearing in the Clebsch-transformation applies to every stationary flow field for which there exist surfaces orthogonal to the streamline

19

system. (Regarding flow regimes for which such orthogonal surfaces do not exist, see, for example, Prásil [Ref. (17)]). The physical meaning of the functions φ, λ, μ pointed out above can be deduced in another way by application of Hamilton's variational principle [Ref. (18)], a topic I will discuss in detail elsewhere in connection with the generalization to the non-stationary case.

It is a pleasure for me to thank my dear friend and colleague Prof. Hilding Köhler of Uppsala, who, in comprehensive discussions over hydrodynamic questions, encouraged me to investigate the present problem.

References

1) A. CLEBSCH, Über die Integration der hydrodynamischen Gleichungen. Crelles Journal, Bd. 56, Berlin 1859, S. 1–10.

2) W. WIEN, Lehrbuch der Hydrodynamik. Leipzig 1900, S. 23–25.

3) H. LAMB, Lehrbuch der Hydrodynamik. 2. Aufl. der deutschen Ausgabe. Leipzig u. Berlin 1931, S. 270–272.

4) V. BJERKNES und Mitarbeiter, Physikalische Hydrodynamik. Berlin 1933, 146.

5) H. LAMB, a. a. O., S. 266.

6) H. LAMB, Hydrodynamics. Sixth Edition, Cambridge 1932, p. 243 f.

7) A. S. RAMSEY, A Treatise on Hydromechanics. Part II. Hydrodynamics. London, 1947, p. 249.

8) H. ERTEL und H. KÖHLER, Ein Theorem über die stationäre Wirbelbewegung kompressibler Flüssigkeiten. ZAMM, Bd. 29, Berlin 1949, S. 109–113.

9) H. LAMB, a. a. O. (Berlin 1931), S. 249; (Cambridge 1932), p. 249.

10) E. MADELUNG, Die mathematischen Hilfsmittel des Physikers. 4. Aufl. Berlin, Göttingen, Heidelberg 1950, S. 364.

11) G. HAMEL, Theoretische Mechanik. Berlin, Göttingen, Heidelberg 1949, S. 317 ff.

12) TH. PÖSCHL, Einführung in die Analytische Mechanik. Karlsruhe 1949, S. 123 f.

13) G. JOOS, Lehrbuch der Theoretischen Physik. 7. Aufl. Leipzig 1950, S. 111.

14) H. ERTEL, Ein Theorem über asynchron-periodische Wirbelbewegungen kompressibler Flüssigkeiten. Miscellanea Acad. Berolinensia, Bd. I. Berlin 1950, S. 62–68.

15) H. ERTEL und C.-G. ROSSBY, Ein neuer Erhaltungssatz der Hydrodynamik. Sitz.-Ber. Dt. Akad. D. Wiss. Berlin, Math.-naturwiss. Kl., 1949, Nr. 1.

16) H. ERTEL and C.-G. ROSSBY, A new Conservation-Theorem of Hydrodynamics. Geofi scia pura e applicata, Milano, Vol. XIV, 1949, p. 189–193.

17) F. PRÁSIL, Technische Hydrodynamik. 2. Aufl. Berlin 1926, S. 75 ff.

18) L. DE BROGLIE, Einführung in die Wellenmechanik. Leipzig 1929, S. 15–17.

6   ERTEL (1955a): Canonical algorithm for hydrodynamic vorticity equations

I. Introduction

The goal of this study is to obtain a general equation that allows derivation of all known hydrodynamic vorticity theorems by specification of a single vector $\mathbf{\Psi}$.

In 1942 H. ERTEL [Ref. 1–3] derived the formula

$$\frac{d}{dt}(\alpha \boldsymbol{\xi} \cdot \nabla \psi) = \alpha \boldsymbol{\xi} \cdot \nabla \left(\frac{d\psi}{dt} \right) + \alpha N(p, \alpha, \psi), \quad (6.1)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

is the material time derivative, $\mathbf{v}$ the velocity vector, $\boldsymbol{\xi}$ the vorticity vector, α the specific volume (reciprocal of density), p the pressure, ψ the for specification of the available scalar function, and $N(p, \alpha, \psi) = (\nabla p \times \nabla \alpha) \cdot \nabla \psi$ the number of p, α, ψ unit cells in the unit volume.

Equation (6.1) came close to this goal, as it permitted the then (1942) known vorticity theorems to be deduced through specification of the scalar function ψ. For instance, in equation (6.1), defining cartesian coordinates x, y, z for ψ results in the vorticity equation components for compressible, baroclinic fluids. In the case of adiabatic or general polytropic changes of state, if we substitute the potential or polytropic temperatures

for ψ, such that

$$\psi = \psi(p, \alpha), \qquad \frac{d\psi}{dt} = 0$$

is valid, the resulting conservation principle that arises from (6.1) [Ref. 4] is

$$\frac{d}{dt}\left(\alpha\boldsymbol{\xi} \cdot \nabla\psi\right) = 0$$

[compare also with Ref. 5–8].

In 1948 a further conservation principle [H. ERTEL and C.-G. ROSSBY, Ref. 9, 10] for barotropic fluids was found for all particles with the same piezotropic relation $\alpha(p)$

$$\frac{d}{dt}\left\{\alpha\boldsymbol{\xi} \cdot (\mathbf{v} - \nabla W)\right\} = 0, \qquad (6.2)$$

in which $W = \int_0^t L\,dt$ is the forcing function (L is the Lagrange function); (6.2) cannot be considered a special case of (6.1), and this fact suggests the generalization shown below.

II. Canonical algorithms for hydrodynamic vorticity equations

To obtain the general formula (the canonical algorithm), which through specification of a vector $\boldsymbol{\Psi}$ yields all known vorticity equations, we start with the hydrodynamic motion equations

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times (2\boldsymbol{\Omega} + \nabla \times \mathbf{v}) \\ = -\nabla\phi - \alpha\nabla p, \qquad (6.3)$$

($\boldsymbol{\Omega}$ = Earth's rotation vector, taken to be constant, ϕ = geopotential), which we write through introduction of the absolute velocity

$$\mathbf{V} = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r} \qquad (6.4)$$

($\mathbf{r}$ = position vector from a point on the axis of rotation to the fluid particle) and the absolute vorticity

$$\boldsymbol{\xi} = \nabla \times \mathbf{V} = \nabla \times \mathbf{v} + 2\boldsymbol{\Omega} \qquad (6.5)$$

in the form

$$\frac{\partial\mathbf{V}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \boldsymbol{\xi} = -\nabla\phi - \alpha\nabla p, \qquad (6.6)$$

from which, through application of the curl operation, follows

$$\frac{\partial\boldsymbol{\xi}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\xi}) = \mathbf{N}(p, \alpha) \qquad (6.7)$$

with the solenoidal vector

$$\mathbf{N}(p, \alpha) = \nabla p \times \nabla\alpha.$$

After easy manipulation, scalar multiplication with the vector $\boldsymbol{\Psi}$ gives

$$\frac{\partial}{\partial t}(\boldsymbol{\xi}\cdot\boldsymbol{\Psi}) = \boldsymbol{\xi}\cdot\frac{\partial\boldsymbol{\Psi}}{\partial t} + \boldsymbol{\Psi}\cdot\nabla\times(\mathbf{v}\times\boldsymbol{\xi}) + \mathbf{N}(p, \alpha)\cdot\boldsymbol{\Psi}. \quad (6.8)$$

We have

$$\boldsymbol{\Psi} \cdot \nabla \times (\mathbf{v} \times \boldsymbol{\xi}) \\ = (\mathbf{v} \times \boldsymbol{\xi}) \cdot \nabla \times \boldsymbol{\Psi} - \nabla \cdot \{\boldsymbol{\Psi} \times (\mathbf{v} \times \boldsymbol{\xi})\} \\ = (\mathbf{v} \times \boldsymbol{\xi}) \cdot \nabla \times \boldsymbol{\Psi} - \nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) - \boldsymbol{\xi}(\mathbf{v} \cdot \boldsymbol{\Psi})\} \\ = (\mathbf{v} \times \boldsymbol{\xi}) \cdot \nabla \times \boldsymbol{\Psi} - \nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\} + \boldsymbol{\xi} \cdot \nabla(\mathbf{v} \cdot \boldsymbol{\Psi}) \\ = \boldsymbol{\xi} \cdot \{\nabla(\mathbf{v} \cdot \boldsymbol{\Psi}) - \mathbf{v} \times \nabla \times \boldsymbol{\Psi}\} - \nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\},$$

so that (6.8) can be written in the form

$$\frac{\partial}{\partial t}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) + \nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\} \\ = \boldsymbol{\xi} \cdot \left\{\frac{\partial\boldsymbol{\Psi}}{\partial t} + \nabla(\mathbf{v} \cdot \boldsymbol{\Psi}) - \mathbf{v} \times \nabla \times \boldsymbol{\Psi}\right\} \qquad (6.9) \\ + \mathbf{N}(p, \alpha) \cdot \boldsymbol{\Psi}.$$

Since

$$\frac{\partial\boldsymbol{\Psi}}{\partial t} + \nabla(\mathbf{v} \cdot \boldsymbol{\Psi}) - \mathbf{v} \times (\nabla \times \boldsymbol{\Psi}) = \frac{D\boldsymbol{\Psi}}{Dt} \qquad (6.10)$$

is that differential operation by which the substantial change of the scalar product of the vector $\boldsymbol{\Psi}$ can be shown [Ref. 11–12] to be a "material change," and using a line element $\mathbf{ds}$ which follows the fluid, such that

$$\frac{d}{dt}(\boldsymbol{\Psi} \cdot \mathbf{ds}) = \frac{D\boldsymbol{\Psi}}{Dt} \cdot \mathbf{ds}, \qquad (6.11)$$

equation (6.9) can be written as

$$\frac{\partial}{\partial t}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) + \nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\} \\ = \boldsymbol{\xi} \cdot \frac{D\boldsymbol{\Psi}}{Dt} + \mathbf{N}(p, \alpha) \cdot \boldsymbol{\Psi}. \qquad (6.12)$$

After multiplying this equation by α and using

$$\nabla \cdot \{\mathbf{v}(\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\} = \mathbf{v} \cdot \nabla(\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) + (\boldsymbol{\xi} \cdot \boldsymbol{\Psi})\nabla \cdot \mathbf{v}$$

and the continuity equation

$$\alpha\nabla \cdot \mathbf{v} = \frac{\partial\alpha}{\partial t} + \mathbf{v} \cdot \nabla\alpha = \frac{d\alpha}{dt}, \qquad (6.13)$$

we obtain the sought after solution

$$\boxed{\frac{d}{dt}(\alpha\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) = \alpha\boldsymbol{\xi} \cdot \frac{D\boldsymbol{\Psi}}{Dt} + \alpha\mathbf{N}(p, \alpha) \cdot \boldsymbol{\Psi},} \qquad (6.14)$$

where on the right side one must still add the term $\alpha(\nabla \times \mathbf{F}) \cdot \boldsymbol{\Psi}$ when the external force $\mathbf{F}$ is not deriveable from a potential ϕ:

$$\boxed{\begin{aligned} \frac{d}{dt}(\alpha\boldsymbol{\xi} \cdot \boldsymbol{\Psi}) = \alpha\boldsymbol{\xi} \cdot \frac{D\boldsymbol{\Psi}}{Dt} \\ + \alpha\{\nabla \times \mathbf{F} + \mathbf{N}(p, \alpha)\} \cdot \boldsymbol{\Psi}. \end{aligned}} \quad (6.15)$$

In the following cases we will always assume there exists a potential of the external force, and thus use the form (6.14) from the established algorithms (6.14, 6.15).

III. Derivation of the hydrodynamic vorticity equation by means of the canonical algorithm by specification of the vector $\boldsymbol{\Psi}$

Equations (6.14) and/or (6.15) are to be understood to be general formulae, which supply a corresponding vorticity equation for each special choice of the vector $\boldsymbol{\Psi}$.

A. If, for example, we set $\boldsymbol{\Psi} = \mathbf{i}$ ($\mathbf{i}$ = the unit vector of the x-axis of a cartesian coordinate system), equation (6.14) supplies, since equation (6.10) is

$$\frac{D\mathbf{i}}{Dt} = \nabla v_x,$$

the x-components of the generalized Helmholtz vorticity equation for compressible and baroclinic fluids

$$\frac{d}{dt}(\alpha\xi_x) = \alpha\boldsymbol{\xi} \cdot \nabla v_x + \alpha N_x(p, \alpha).$$

One can obtain the y and z-components from equation (6.14) by choosing $\boldsymbol{\Psi} = \mathbf{j}$ and $\boldsymbol{\Psi} = \mathbf{k}$ respectively ($\mathbf{j}, \mathbf{k}$ are unit vectors in the y and z directions). [Compare, for example, Ref. 13].

B. If one sets $\boldsymbol{\Psi} = \nabla\psi$ (ψ is a scalar function) in equation (6.14) in accordance with equation (6.10), we have

$$\begin{aligned} \frac{D}{Dt}\nabla\psi &= \frac{\partial}{\partial t}\nabla\psi + \nabla(\mathbf{v} \cdot \nabla\psi) \\ &= \nabla\left(\frac{\partial\psi}{\partial t} + \mathbf{v} \cdot \nabla\psi\right) = \nabla\left(\frac{d\psi}{dt}\right) \end{aligned}$$

and equation (6.1) will result from equation (6.14).

C. If we set $\boldsymbol{\Psi} = \mathbf{v} - \nabla W$ in equation (6.14) with the Lagrange function

$$L = \frac{dW}{dt} = \frac{v^2}{2} - \left(\phi + \int \alpha\, dp\right)$$

(nonrotating system $\boldsymbol{\Omega} = 0$ and autobarotropic $\alpha = \alpha(p)$), then equation (6.10) will be

$$\begin{aligned} &\frac{D}{Dt}(\mathbf{v} - \nabla W) \\ &= \frac{\partial\mathbf{v}}{\partial t} - \nabla\frac{\partial W}{\partial t} + \nabla(v^2 - \mathbf{v} \cdot \nabla W) \\ &\quad - \mathbf{v} \times \nabla \times \mathbf{v} \\ &= \frac{\partial\mathbf{v}}{\partial t} + \nabla v^2 - \mathbf{v} \times \nabla \times \mathbf{v} \\ &\quad - \nabla\left(\frac{\partial W}{\partial t} + \mathbf{v} \cdot \nabla W\right) \\ &= \frac{\partial\mathbf{v}}{\partial t} + \nabla v^2 - \mathbf{v} \times \nabla \times \mathbf{v} - \nabla\left(\frac{dW}{dt}\right) \\ &= \frac{\partial\mathbf{v}}{\partial t} + \nabla v^2 - \mathbf{v} \times \nabla \times \mathbf{v} \\ &\quad - \nabla\left\{\tfrac{1}{2}v^2 - \left(\Phi + \int \alpha\, dp\right)\right\} \\ &= \frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \nabla \times \mathbf{v} \\ &\quad + \nabla\left(\Phi + \int \alpha\, dp\right) = 0 \end{aligned}$$

since the hydrodynamic motion equation in this case is

$$\begin{aligned} \frac{\partial\mathbf{v}}{\partial t} &+ \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \nabla \times \mathbf{v} \\ &= -\nabla\left(\Phi + \int \alpha\, dp\right) \end{aligned}$$

and the conservation principle (6.2) with $\boldsymbol{\xi} = \nabla \times \mathbf{v}$ will result from equation (6.14). In the case of a rotating system, in order to derive the conservation principle (6.2), in which then $\boldsymbol{\xi}$ represents the absolute vorticity $2\boldsymbol{\Omega} + \nabla \times \mathbf{v}$, and $\mathbf{v}$ is replaced by the absolute velocity,

$$\mathbf{V} = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}$$

and the Lagrange function [Ref. 14] is

$$\begin{aligned} \overset{*}{L} &= \frac{d\overset{*}{W}}{dt} = \tfrac{1}{2}v^2 + \boldsymbol{\Omega} \cdot (\mathbf{r} \times \mathbf{v}) \\ &= -\left(\Phi + \int \alpha\, dp\right) \end{aligned}$$

in equation (6.14) one must now use $\boldsymbol{\Psi} = \mathbf{V} - \nabla \overset{*}{W}$. Then according to (6.10)

$$\frac{D}{Dt}\left(\mathbf{V} - \nabla \overset{*}{W}\right) = 0$$

based on the hydrodynamic motion equation (6.3) and the conservation principle (6.2) [Ref. 9, 10] we have

$$\frac{d}{dt}\left\{\alpha\boldsymbol{\xi} \cdot (\mathbf{V} - \nabla \overset{*}{W})\right\} = 0$$

with $\boldsymbol{\xi} = \nabla \times \mathbf{V}$ (absolute vorticity).

IV. Summary

We have obtained a general formula (canonical algorithm) that permits derivation of all presently known hydrodynamic vorticity equations in differential form through specification of a single vector.

References

[1] H. ERTEL, Ein neuer hydrodynamischer Wirbelsatz. Meteorol. Zeitschr. (Braunschweig) **1942**, 277–281.

[2] H. ERTEL, Über das Verhältnis des neuen hydrodynamischen Wirbelsatzes zum Zirkulationssatz von V. BJERKNES. Ebenda, 385–387.

[3] H. ERTEL, Über hydrodynamische Wirbelsätze. Physikal. Zeischr. **1942**, 526–529.

[4] H. ERTEL, Ein neuer hydrodynamischer Erhaltungassatz. Die Naturwissenschaften (Berlin) **1942**, 543–544.

[5] E. KLEINSCHMIDT jr., Über Aufbau und Entestehung von Zyklonen. Meteorol. Rundschau **1950**, 1–6, 54–61; **1951**, 89–96.

[6] F. BAUR, Linke's Meteorol. Taschenbuch. Neue Ausgabe. H.Bd. Leipzig **1953**, 295–298 (P. RAETHJEN).

[7] K. OSWATITSCH, Gasdynamik. Wien **1952**, 156.

[8] N.J. KOTSCHIN, I.A. KIBEL und N.W. ROSE, Theoretische Hydromechanik. Bd. I. Berlin **1954**, 501 (Deutsche Ausgabe, red. von K. KRIENES).

[9] H. ERTEL und C.-G. ROSSBY, A new conservation-theorem of hydrodynamics. Geosfi sica pura e applicata. Milano **1948**, Vol. XIV, Fasc. 3–4.

[10] H. ERTEL und C.-G. ROSSBY, Ein neuer Erhaltungs-Satz der Hydrodynamik. Sitz.-Ber. Dt. Akad. d. Wiss. Berlin, Kl. f. Math. u. allgem. Naturwiss., **1949**, Nr.1.

[11] W.v. IGNATOWSKY, Die Vektoranalysis und ihre Anwendungen in der theoretischen Physik. Bd. 11. Leipzig-Berlin **1910**, 62.

[12] E. LOHR, Vektor- und Dyadenrechnung für Physiker und Techniker. 2. Aufl. Berlin **1950**, 315–317.

[13] H. LAMB, Hydrodynamics. First American Edition New York **1945**, 205.

[14] H. ERTEL, Methoden und Probleme der Dynamischen Meteorologie. Berlin **1938** (Ann Arbor, Mich. **1943**), 41–44.

7   ERTEL (1955b): A new hydrodynamical vorticity theorem

I. Introduction

If the motion of a compressible fluid relative to the rotating earth is written through the hydrodynamic vector equation

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\tfrac{1}{2} v^2 \right) - \mathbf{v} \times (2\boldsymbol{\Omega} + \nabla \times \mathbf{v}) \\ = -\nabla \Phi - \alpha \nabla p, \tag{7.1}$$

where $\mathbf{v}$ is the velocity vector relative to the Earth, v the absolute value of the velocity vector, $\boldsymbol{\Omega}$ the constant rotation vector of the Earth, Φ the geopotential, α the specific volume (reciprocal of density ρ), and p the pressure, then the time change of circulation relative to the Earth

$$\frac{D}{Dt} \oint v_s \, ds = \frac{D}{Dt} \iint (\nabla \times \mathbf{v})_n \, dF \tag{7.2}$$

is a substantial one, moving with the fluid, of the closed curve (a line element of which $= ds$) or of the vorticity flux through every material surface (an element of which $= dF$, $\mathbf{n}$ = positive normal), which has a closed material curve as the boundary; this time change is determined from the circulation equation of V. Bjerknes [Ref. (1–3)] through the equation

$$\frac{D}{Dt} \left(\oint v_s \, ds + 2\Omega F_\Omega \right) = N(p, \alpha) \tag{7.3}$$

in which

$$F_\Omega = \iint \cos(\mathbf{n}, \boldsymbol{\Omega}) \, dF \tag{7.4}$$

represents the projection of the surface $\iint dF$ onto any latitudinal plane (e.g., the equatorial plane) and where

$$N(p, \alpha) = \iint (\nabla p \times \nabla \alpha)_n \, dF \tag{7.5}$$

represents the number of isobar-isostern unit solenoids in the surface $\iint dF$.

If one introduces the absolute vorticity

$$\boldsymbol{\xi} = 2\boldsymbol{\Omega} + \nabla \times \mathbf{v} \tag{7.6}$$

for which flux through $\iint dF$, the following holds:

$$\iint \xi_n \, dF = \oint v_s \, ds + 2\Omega F_\Omega \qquad (7.7)$$

then the circulation equation of V. Bjerknes can be written

$$\frac{D}{Dt} \iint \xi_n \, dF = \iint (\nabla p \times \nabla \alpha)_n \, dF. \qquad (7.8)$$

The operator D/Dt which describes the time change of $\iint a_n \, dF$ for a vector $\mathbf{a}$ through a moving material surface $\iint dF$ means explicitly the operation

$$\frac{D}{Dt} \iint a_n \, dF$$
$$= \iint \left\{ \frac{\partial \mathbf{a}}{\partial t} + \nabla \times (\mathbf{a} \times \mathbf{v}) + \mathbf{v} \nabla \cdot \mathbf{a} \right\}_n dF, \qquad (7.9)$$

[Ref. (4,5)] which for $\mathbf{a} = \boldsymbol{\xi}$ with $\nabla \cdot \boldsymbol{\xi} = 0$, immediately produces the V. Bjerknes circulation equation (7.8) in that the curl of (7.1), with reference to (7.6), results in

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\boldsymbol{\xi} \times \mathbf{v}) = \nabla p \times \nabla \alpha. \qquad (7.10)$$

The resulting vorticity theorem differs from the V. Bjerknes circulation equation in that, in place of the flux of the vorticity vector $\iint \xi_n \, dF$ through a material surface, one looks at the generalized flux $\iint \psi \xi_n \, dF$, where ψ denotes a scalar function $\psi(x, y, z, t)$ of the space coordinates x, y, z and the time t.

II. Derivation of the new vorticity theorem

We first multiply (7.10) by ψ to obtain

$$\psi \frac{\partial \boldsymbol{\xi}}{\partial t} + \psi \nabla \times (\boldsymbol{\xi} \times \mathbf{v}) = \psi(\nabla p \times \nabla \alpha). \qquad (7.11)$$

Because

$$\psi \nabla \times (\boldsymbol{\xi} \times \mathbf{v}) = \nabla \times (\psi \boldsymbol{\xi} \times \mathbf{v}) + \nabla \psi \times (\mathbf{v} \times \boldsymbol{\xi}) \quad (7.12)$$

and

$$\nabla \psi \times (\mathbf{v} \times \boldsymbol{\xi}) = \mathbf{v}(\boldsymbol{\xi} \cdot \nabla \psi) - \boldsymbol{\xi}(\mathbf{v} \cdot \nabla \psi), \qquad (7.13)$$

we can write

$$\psi \frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\psi \boldsymbol{\xi} \times \mathbf{v}) + \mathbf{v}(\boldsymbol{\xi} \cdot \nabla \psi) - \boldsymbol{\xi}(\mathbf{v} \cdot \nabla \psi)$$
$$= \psi(\nabla p \times \nabla \alpha). \qquad (7.14)$$

If we write

$$\psi \frac{\partial \boldsymbol{\xi}}{\partial t} = \frac{\partial (\psi \boldsymbol{\xi})}{\partial t} - \boldsymbol{\xi} \frac{\partial \psi}{\partial t}, \qquad (7.15)$$

and introduce the Eulerian material derivative operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \qquad (7.16)$$

and if we also observe that

$$\boldsymbol{\xi} \cdot \nabla \psi = \nabla \cdot (\psi \boldsymbol{\xi}) \qquad (7.17)$$

(since $\nabla \cdot \boldsymbol{\xi} = 0$), the result from (7.14) is the equation

$$\frac{\partial (\psi \boldsymbol{\xi})}{\partial t} + \nabla \times (\psi \boldsymbol{\xi} \times \mathbf{v}) + \mathbf{v} \nabla \cdot (\psi \boldsymbol{\xi}) - \boldsymbol{\xi} \frac{d\psi}{dt}$$
$$= \psi(\nabla p \times \nabla \alpha). \quad (7.18)$$

If the subscript n denotes the normal to a surface element of a unbounded material surface $\iint dF$, then from (7.18) it follows that

$$\iint \left\{ \frac{\partial (\psi \boldsymbol{\xi})}{\partial t} + \nabla \times (\psi \boldsymbol{\xi} \times \mathbf{v}) + \mathbf{v} \nabla \cdot (\psi \boldsymbol{\xi}) \right\}_n dF$$
$$- \iint \xi_n \frac{d\psi}{dt} dF = \iint \psi(\nabla p \times \nabla \alpha)_n dF, \quad (7.19)$$

or, with the aid of (7.9),

$$\boxed{\begin{aligned} \frac{D}{Dt} \iint \psi \xi_n dF &- \iint \xi_n \frac{d\psi}{dt} dF \\ &= \iint \psi(\nabla p \times \nabla \alpha)_n dF. \end{aligned}} \qquad (7.20)$$

This equation represents the new vorticity theorem, which hereby has been derived from the hydrodynamic vector equation (7.1), with the introduction of the specialized scalar function ψ.

III. Special cases

From the vorticity theorem (7.20), special cases can be derived by specification of the scalar function ψ, so that the open surface $\iint dF$ can be made to be a closed surface.

Examples:

a) V. Bjerknes' circulation theorem (7.8) can be obtained as a special case of the vorticity theorem (7.20) for

$$\psi = \text{const.} \qquad (7.21)$$

b) Let us choose

$$\psi = T^{-1} \left(\frac{p}{p_0} \right)^\kappa = \theta^{-1} \qquad (7.22)$$

24

(θ is the potential temperature, p_0 is the reference pressure, $\gamma = c_p/c_v$, $\kappa = (\gamma - 1)/\gamma$) and consider adiabatic flow so that

$$\frac{d\theta}{dt} = 0. \qquad (7.23)$$

From (7.20) we then obtain

$$\frac{D}{Dt} \iint \frac{\xi_n}{\theta} dF = \iint \theta^{-1} (\nabla p \times \nabla \alpha)_n dF. \quad (7.24)$$

The right hand side of this equation can be written as the flux of a solenoidal vector, since it follows from the equation of state for an ideal gas ($p\alpha = RT$, $R = c_p - c_v$ is the gas constant, T is the absolute temperature) that

$$\theta = \frac{p_0^\kappa}{R} p^{\frac{1}{\gamma}} \alpha, \qquad (7.25)$$

and therefore that

$$\theta^{-1}(\nabla p \times \nabla \alpha) = \nabla \left(\frac{p}{p_0}\right)^\kappa \times \nabla(c_p \ln \alpha). \quad (7.26)$$

Consequently we have

$$\frac{D}{Dt} \iint \frac{\xi_n}{\theta} dF$$
$$= \iint \left\{ \nabla \left(\frac{p}{p_0}\right)^\kappa \times \nabla(c_p \ln \alpha) \right\}_n dF \quad (7.27)$$

with the solenoidal vector

$$\nabla \left(\frac{p}{p_0}\right)^\kappa \times \nabla(c_p \ln \alpha) = \nabla \left(\frac{1}{\theta}\right) \times \nabla(c_p T). \quad (7.28)$$

Also the choice $\psi = \theta$ (instead of $\psi = \theta^{-1}$) gives the flux of a solenoidal vector for the right hand side of (7.20).

c) Generally, the following holds

$$\frac{D}{Dt} \iint J \xi_n dF = \iint \left\{ \nabla f(p) \times \nabla g(\alpha) \right\}_n dF \quad (7.29)$$

for every hydrodynamic invariant J ($dJ/dt = 0$) that allows itself to be represented in the form

$$J = \frac{df(p)}{dp} \frac{dg(\alpha)}{d\alpha} \qquad (7.30)$$

(for example the polytropic temperature during polytropic changes of state).

Suppose one chooses in the vorticity theorem (7.20) a closed surface $S = \iint ds$ for $\iint dF$; this surface encloses the volume $V = \iiint dV$. Since the application of

Gauss' theorem yields

$$\iint \psi(\nabla p \times \nabla \alpha)_n dS = \iiint (\nabla p \times \nabla \alpha) \cdot \nabla \psi \, dV$$
$$= N(p, \alpha, \psi), \qquad (7.31)$$

the following equation results:

$$\frac{D}{Dt} \iint \psi \xi_n \, dS - \iint \xi_n \frac{d\psi}{dt} \, dS = N(p, \alpha, \psi). \quad (7.32)$$

This is the integral form of the vorticity equation derived by the author in 1942 [Ref. (6)], where $N(p, \alpha, \psi)$ denotes the number of (p, α, ψ) unit cells enclosed in the volume V. On the other hand, the vorticity theorem (7.20) can be understood as the generalization of (7.32) for the case of an unbounded surface integration.

IV. Summary

A hydrodynamic vorticity theorem in a form compatible with the fundamental equation of hydrodynamics for compressible fluids has been derived, from which integral vorticity equations, for example the Bjerknes circulation theorem, can be represented in a simple form through specification of a scalar function.

References

[1] V. BJERKNES, Über einen hydrodynamischen Zirkulationssatz und seine Anwendung auf die Mechanik der Atmosphäre und des Weltmeeres. Kongl. Svenska Vetenskapsakad. Handl. **31** (1908). — Das dynamische Prinzip der Zirkulationsbewegungen in der Atmosphäre. Meteorol. Zeitschr. **1900**, 97–106. — Zirkulation relativ zu der Erde. Ebenda **1902**, 96–108.

[2] V. BJERKNES und Mitarbeiter, Physikalische Hydrodynamik. Berlin **1933**, 146.

[3] J. HOLMBOE, G. F. FORSYTHE and W. GUSTIN, Dynamic Meteorology. New York and London **1945**, 295–318.

[4] E. MADELUNG, Die mathematischen Hilfsmittel des Physikers. 4. Aufl. Berlin-Göttingen-Heidelberg **1950**, 203.

[5] R. GANS, Vector Analysis with Applications to Physics. London and Glasgow **1950**, 59–60.

[6] H. ERTEL, Über hydrodynamische Wirbelsätze. Physikalische Zeitschrift, **1942**, 526–529.

8 ERTEL (1956): Orthogonal trajectory systems in stationary two-dimensional flow of an ideal incompressible fluid

I. Remarks on methods for the transformation of the hydrodynamic equations

A planar flow field is characterized by the property that the flow of each and every fluid particle evolves in a plane and the flow in each and every plane is identical. The entire flow field is determined by the flow in one plane, say the x, y-plane [M. LAGALLY, L. M. MILNE-THOMSON, Ref. (1,2)].

In the absence of exterior forces and for stationary motion of an ideal fluid, the Eulerian hydrodynamical equations take the matrix form

$$\begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\frac{1}{\rho} \begin{pmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \end{pmatrix} \quad (8.1)$$

where v_x, v_y are the wind components, p the pressure, and ρ the density.

The flow is assumed to be incompressible, so that the trace of the 2×2 matrix of the velocity derivatives

$$M = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{pmatrix} \quad (8.2)$$

vanishes, i.e.,

$$M_{xx} + M_{yy} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (8.3)$$

(continuity equation).

From the matrix M and its transpose

$$M' = \begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_y}{\partial x} \\ \dfrac{\partial v_x}{\partial y} & \dfrac{\partial v_y}{\partial y} \end{pmatrix} \quad (8.4)$$

we can obtain the antisymmetric matrix

$$M - M' = \begin{pmatrix} 0 & \dfrac{\partial v_x}{\partial y} - \dfrac{\partial v_y}{\partial x} \\ \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\zeta \\ +\zeta & 0 \end{pmatrix} \quad (8.5)$$

where

$$\zeta = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (8.6)$$

is the vorticity. We now utilize a common transformation of the hydrodynamical equations through the substitution

$$M_{xy} = \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} - \zeta,$$
$$M_{yx} = \frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial y} + \zeta \quad (8.7)$$

for the off-diagonal elements of the matrix of velocity derivatives (8.2); this substitution with $v^2 = v_x^2 + v_y^2$ yields

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{v^2}{2}\right) - v_y\zeta = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial}{\partial y}\left(\dfrac{v^2}{2}\right) + v_x\zeta = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \end{cases} \quad (8.8)$$

for the hydrodynamical equations.

In the following discussion we make use of another transformation and note that the matrix elements on the main diagonal in (8.2), i.e., the elements

$$M_{xx} = \frac{\partial v_x}{\partial x}, \qquad M_{yy} = \frac{\partial v_y}{\partial y},$$

can be written, with the aid of the continuity equation (8.3), as

$$M_{xx} = -\frac{\partial v_y}{\partial y}, \qquad M_{yy} = -\frac{\partial v_x}{\partial x}, \quad (8.9)$$

so that the matrix (8.2) becomes

$$M = \begin{pmatrix} -\dfrac{\partial v_y}{\partial y} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & -\dfrac{\partial v_x}{\partial x} \end{pmatrix}. \quad (8.10)$$

II. Orthogonality of isogons and isobars for non-unidirectional, stationary, planar flow of an ideal, incompressible fluid

For inhomogeneous flow fields the matrices (8.2) and (8.10) are not null-matrices, and with the matrix (8.10) the hydrodynamical momentum equations are

$$\begin{cases} -v_x\dfrac{\partial v_y}{\partial y} + v_y\dfrac{\partial v_x}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ +v_x\dfrac{\partial v_y}{\partial x} - v_y\dfrac{\partial v_x}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}. \end{cases} \quad (8.11)$$

We define the direction of the wind vector $\mathbf{v} = (v_x, v_y)$ as the angle with respect to the positive x-axis, so that

$$v_x = v\cos\theta, \quad v_y = v\sin\theta, \quad \frac{v_y}{v_x} = \tan\theta. \quad (8.12)$$

The wind direction field $\theta(x, y)$ can be illustrated through the isolines $\theta(x, y) = \text{const} (= c_1, c_2, \cdots)$, which are points with equal values of wind direction; these isolines can be constructed graphically from the streamline field [see, for example, J. W. SANDSTRÖM, W. HORT, F. E. RELTON, Ref. (3–5)]. In dynamic meteorology the lines $\theta(x, y) = \text{const}$ are referred to as "isogons." We exclude the special case in which the function $\theta(x, y)$ has the same value at all field points, and hence only treat the case of non-unidirectional flow fields.

We now have

$$\begin{cases} -v_x\dfrac{\partial v_y}{\partial y} + v_y\dfrac{\partial v_x}{\partial y} = -v_x^2\dfrac{\partial}{\partial y}\left(\dfrac{v_y}{v_x}\right) = -v^2\dfrac{\partial\theta}{\partial y}, \\[3mm] +v_x\dfrac{\partial v_y}{\partial x} - v_y\dfrac{\partial v_x}{\partial x} = +v_x^2\dfrac{\partial}{\partial x}\left(\dfrac{v_y}{v_x}\right) = +v^2\dfrac{\partial\theta}{\partial x}, \end{cases} \quad (8.13)$$

and the hydrodynamical momentum equations (8.11) take the form:

$$\begin{cases} +\rho v^2\dfrac{\partial\theta}{\partial y} = +\dfrac{\partial p}{\partial x}, \\[3mm] -\rho v^2\dfrac{\partial\theta}{\partial x} = +\dfrac{\partial p}{\partial y}, \end{cases} \quad (8.14)$$

so that

$$\boxed{\frac{\partial\theta}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial p}{\partial y} = 0,} \quad (8.15)$$

and we have the following result:

In the planar flow of an ideal incompressible fluid the isogons and isobars form an orthogonal trajectory system in the stationary case.

In accordance with its derivation, this statement applies to both planar rotational flows as well as planar potential flows, for the acceptance of the existence of a velocity potential was not required; exempted from the statement are the the the singular points of the function $\theta(x, y)$ [R. ZURMÜHL, Ref. (6)].

III. Orthogonality of isogons and isolines of potential energy when there exists a potential for external forces

If X, Y denote the components of the external force

$$(X, Y) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi,$$

expressed in terms of the potential Φ, (8.15) changes to a form in which the pressure p is replaced by the potential energy $E_{\text{pot}} = \rho\Phi + p$:

$$\boxed{\frac{\partial\theta}{\partial x}\frac{\partial E_{\text{pot}}}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial E_{\text{pot}}}{\partial y} = 0,} \quad (8.16)$$

and the hydrodynamical momentum equations (8.14) can be written as

$$\begin{cases} \rho v^2\dfrac{\partial\theta}{\partial y} = \dfrac{\partial}{\partial x}(\rho\Phi + p), \\[3mm] -\rho v^2\dfrac{\partial\theta}{\partial x} = \dfrac{\partial}{\partial y}(\rho\Phi + p). \end{cases} \quad (8.17)$$

IV. Orthogonality of isogons and isotachs in stationary, two-dimensional, potential flow

Suppose there exists a potential flow:

$$(v_x, v_y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\varphi, \qquad \zeta \equiv 0, \quad (8.18)$$

so that in the stationary case the Bernoulli equation is

$$\rho\frac{v^2}{2} + E_{\text{pot}} = \rho\frac{v^2}{2} + \rho\Phi + p = \text{const.} \quad (8.19)$$

Eliminating E_{pot} from (8.16) with the aid of the Bernoulli equation, we obtain

$$\boxed{\frac{\partial\theta}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial v}{\partial y} = 0.} \quad (8.20)$$

The curves $v(x, y) = \text{const} (= v_1, v_2, \cdots)$ are the isotachs. *In non-unidirectional, stationary, planar, potential flow of an ideal incompressible fluid the isogons and isotachs must form an orthogonal trajectory system.*

V. Unidirectional, planar, stationary potential flow

In the case that $\theta(x, y) = \text{const}$ in the entire field, then $\partial\theta/\partial x = \partial\theta/\partial y = 0$, so that the hydrodynamical equations (8.17) yield: $\rho\Phi + p = \text{const}$; with the aid of the Bernoulli equation (8.19), it follows that: $v = \text{const}$, and the potential flow is homogeneous.

VI. Conclusions

The construction of isogons for non-unidirectional, stationary, planar flow of an ideal incompressible fluid is made possible when only the streamlines (and not necessarily the streamfunction) are given, so that the isogons are oriented perpendicular to

a) *the pressure field*, if no other force field exists,

b) *the potential energy field*, if another force field exists,

c) *the isotach field*, if a potential flow exists,

so that the isolines in each case form an orthogonal trajectory system with the isogons.

For a given streamfunction, the streamlines are determined; then the isolines in each case (i.e., pressure, potential energy, or isotachs) can also be determined because in each case their gradient is completely determined.

References

1. M. LAGALLY, Ideale Flüssigkeiten. Handbuch der Physik (Geiger-Scheel), Bd. VII, Berlin 1927, S. 3.

2. L. M. MILNE-THOMSON, Theoretical Hydrodynamics. Third Edition, London 1955, pag. 100.

3. J. W. SANDSTRÖM, Über die Bewegung der Flüssigkeiten. Annalen der Hydrographie und Maritimen Meteorologie, Berlin 1909, S. 242.

4. W. HORT, Die Differentialgleichungen des Ingenieurs. 2. Aufl. Berlin 1925, S. 222.

5. F. E. RELTON, Applied Differential Equations. London and Glasgow 1948, pag. 100.

6. R. ZURMÜHL, Praktische Mathematik für Ingenieure und Physiker. Berlin-Göttingen-Heidelberg 1953, S. 324.

9 ERTEL (1957): A general relationship between wind velocity and the hydrodynamical field intensity in the atmosphere

Summary

A general relationship between the atmospheric wind velocity and the hydrodynamical field intensity can be obtained if we take into account that the acceleration vector is always located in the osculator plane of the trajectory. The relationship is easily obtained from the dynamical condition in the direction of the binormal unit vector of the moving trihedron. The same equation is obtained for the geostrophic wind, the only difference being the meaning of the two unit perpendicular vectors. This equation is valid for all frictionless atmospheric flows.

1. Introduction

In atmospheric mechanics, or dynamical meteorology, which studies the movements of the atmosphere's air masses, we deal fundamentally with the velocity field ($\mathbf{v}$). If the air moves without friction, this field is related to the vectorial field $\mathbf{F}$ of the forces of gravity ($-\nabla\Phi$) and pressure gradient ($-\frac{1}{\rho}\nabla p$) per unit mass, that is,

$$\mathbf{F} = -\nabla\Phi - \frac{1}{\rho}\nabla p,$$

through the Coriolis force

$$\mathbf{C} = 2\mathbf{v} \times \boldsymbol{\omega}$$

and the acceleration

$$\frac{d\mathbf{v}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v}$$

of the particles.

Euler's equations of motion, which can be summarized in a single vectorial equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} - 2\mathbf{v} \times \boldsymbol{\omega} = -\nabla\Phi - \frac{1}{\rho}\nabla p,$$

yield the general relationship between the three vectors, $d\mathbf{v}/dt$, $\mathbf{C}$, and $\mathbf{F}$. Since $\mathbf{F} = \mathbf{0}$ (null vector) should be the equilibrium condition, that is, the case of a hydrostatic field, the force $\mathbf{F} \neq 0$ is referred to as the intensity of the hydrodynamical field (in German: hydrodynamische Feldstärke).

In order to close the system of equations that determine the atmospheric motion, we must add the continuity equation and a thermodynamical relationship. However, we know that this system of equations is among the most complex systems in theoretical physics, because the equations are nonlinear differential equations.

Nevertheless, it so happens in dynamical meteorology that a number of air motion problems can be studied in an approximate sense, if one assumes that the acceleration is always small. An example of this is the geostrophic wind (see, for instance, P. CARRASCO, Meteorología, México 1945, pages 175–176; D. BRUNT, Climatología, Buenos Aires-México 1948, pages 194–202; S. PETTERSSEN, Introducción a la Meteorología, Buenos Aires-México 1951, pages 156–161).

2. The problem

Let us consider a motion in the horizontal plane. We fix a rectangular coordinate system (x, y, z), whose

positive z-axis is vertical. Then $\mathbf{v} = (v_x, v_y, 0)$. We assume that in the fundamental hydrodynamical equations the acceleration can be neglected when compared to the Coriolis force. Now the horizontal motion can be expressed as

$$-2\omega_z v_y = F_x, \qquad +2\omega_z v_x = F_y,$$

if we adopt the right-hand rule as the direct or positive trihedron. Let us rotate the coordinate axes trihedron around the $+z$ axis until the $+x$ axis coincides with the wind direction, that is,

$$\mathbf{v} = (v_x = v = |\mathbf{v}|, 0, 0).$$

As a result, we have

$$v = \frac{1}{2}\frac{F_y}{\omega_z},$$

which is nothing but the well-known formula for the geostrophic wind velocity, where $\omega_z = |\boldsymbol{\omega}|\cos(\pi/2 - \varphi) = \omega\sin\varphi$ is the vertical component of the Earth's angular velocity at the northern latitude φ. Using the unit vectors $\mathbf{j}, \mathbf{k}$ in the y, z directions, respectively, we can write the previous equation as

$$v = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{j})}{(\boldsymbol{\omega} \cdot \mathbf{k})}.$$

Now let us consider the following problem. Given the intensity of the hydrodynamic field $\mathbf{F}$ at a certain point in the atmosphere (and we have yet to introduce in the calculation a rigorous treatment of the influence of acceleration) is it possible to determine the wind velocity $v = |\mathbf{v}|$ at that point with the help of an equation that is similar to the equation obtained for the case of the geostrophic wind?

3. Solution

First we need to study in more detail the acceleration in Euler's equations of motion. The d/dt operator represents a derivative following the motion of the particle trajectories, and if $ds = vdt$ is an infinitesimal distance along the trajectory, then

$$\frac{d\mathbf{v}}{dt} = v\frac{d\mathbf{v}}{ds}.$$

Moreover, let $\mathbf{T}$ be the unit tangent in the direction of the motion of the particle, that is, $\mathbf{v} = \mathbf{T}v$. Euler's vector equation of motion is therefore

$$\mathbf{T}\frac{d}{ds}\left(\frac{v^2}{2}\right) + v^2\frac{d\mathbf{T}}{ds} + 2v\boldsymbol{\omega} \times \mathbf{T} = \mathbf{F}.$$

After a scalar product of this equation by the following vectorial product

$$\mathbf{L} = \frac{\mathbf{T} \times \dfrac{d\mathbf{v}}{ds}}{v\left|\dfrac{d\mathbf{T}}{ds}\right|} = \frac{\mathbf{T} \times \dfrac{d\mathbf{T}}{ds}}{\left|\dfrac{d\mathbf{T}}{ds}\right|}$$

we obtain

$$2v(\boldsymbol{\omega} \cdot \mathbf{M}) = (\mathbf{F} \cdot \mathbf{L}),$$

where $\mathbf{M}$ is the vector

$$\mathbf{M} = \mathbf{T} \times \mathbf{L}.$$

We then effectively have the relationship

$$v = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{L})}{(\boldsymbol{\omega} \cdot \mathbf{M})}$$

which has the form of the solution needed because, as we will show below, the right hand side of the equation obeys all of the conditions established. This is exactly the same equation as the equation for the geostrophic wind. The only difference stems from the meaning of $\mathbf{L}$ and $\mathbf{M}$.

It is immediately apparent that $\mathbf{L}$ is in the normal plane, that is, in the plane perpendicular to $\mathbf{T}$. $\mathbf{M}$ is also in this plane:

$$\mathbf{L} \cdot \mathbf{T} = 0, \qquad \mathbf{M} \cdot \mathbf{T} = 0.$$

Moreover $\mathbf{L}$ is orthogonal to $\mathbf{M} = \mathbf{T} \times \mathbf{L}$:

$$\mathbf{L} \cdot \mathbf{M} = 0,$$

and both are unit vectors:

$$|\mathbf{L}| = |\mathbf{M}| = 1.$$

Finally, we are ready to find the directions of the unit vectors $\mathbf{L}$ and $\mathbf{M}$ in the normal plane with the help of a moving trihedron of three perpendicular unit vectors, namely: the unit tangent $\mathbf{T}$, the binormal unit $\mathbf{B}$ and the principal normal unit $\mathbf{N}$. Defining $\mathbf{T}, \mathbf{B}, \mathbf{N}$ as the right-handed trihedron, we can write

$$\mathbf{T} = \mathbf{B} \times \mathbf{N}, \qquad \mathbf{B} = \mathbf{N} \times \mathbf{T}, \qquad \mathbf{N} = \mathbf{T} \times \mathbf{B},$$

and introducing a well-known equation from the differential geometry

$$\frac{d\mathbf{T}}{ds} = -\kappa\mathbf{N}$$

where κ is the trajectory curvature, we have

$$\left|\frac{d\mathbf{T}}{ds}\right| = \kappa, \qquad \mathbf{T} \times \frac{d\mathbf{T}}{ds} = \kappa\mathbf{B},$$

that is

$$\mathbf{L} = \mathbf{B} \qquad \text{and} \qquad \mathbf{M} = \mathbf{N}$$

(see, for instance, D. J. STRUIK, Geometrí a Diferencial Clássica, Madrid 1955, pages 17–23; F. J. GUILLÉN, Cálculo Tensorial, México 1957, pages 34–38).

The general relationship between the wind velocity and the intensity of the hydrodynamical field in the atmosphere becomes:

$$\boxed{v = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{B})}{(\boldsymbol{\omega} \cdot \mathbf{N})}}$$

or, if we introduce cartesian coordinates,

$$v = \frac{1}{2}\frac{(F_x B_x + F_y B_y + F_z B_z)}{(\omega_x N_x + \omega_y N_y + \omega_z N_z)}$$

from which we see that the geostrophic wind is a special case, for instance,

$$\mathbf{B} = (0, 1, 0), \qquad \mathbf{N} = (0, 0, 1).$$

4. Additional considerations

It therefore seems that the real wind velocity is determined by six parameters, the components of the vectors $\mathbf{B} = (B_x, B_y, B_z)$ and $\mathbf{N} = (N_x, N_y, N_z)$. It is important to point out that there are three relationships among them:

$$B_x^2 + B_y^2 + B_z^2 = 1,$$

$$N_x^2 + N_y^2 + N_z^2 = 1,$$

$$B_x N_x + B_y N_y + B_z N_z = 0,$$

and therefore only three of those parameters are completely independent, a number which is equal to the number of components of the acceleration $d\mathbf{v}/dt = (dv_x/dt, dv_y/dt, dv_z/dt)$. It seems at first sight that the set of three independent parameters can be considered directly equivalent to the acceleration. However, a more careful review will show that this is not the case. We will write the general relationship as

$$v = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{B})}{(\boldsymbol{\omega} \cdot \mathbf{N})} = \frac{1}{2}\frac{\mathbf{F} \cdot \left(\mathbf{T} \times \dfrac{d\mathbf{v}}{dt}\right)}{\boldsymbol{\omega} \cdot \left(\mathbf{T} \times \left(\mathbf{T} \times \dfrac{d\mathbf{v}}{dt}\right)\right)}.$$

This equation yields a relationship between those magnitudes; however, according to a kinematics theorem, the three vectors $d\mathbf{v}/dt$, $\mathbf{T}$ and $\mathbf{N}$ are linearly dependent:

$$\frac{d\mathbf{v}}{dt} = \mathbf{T}\left(\frac{d\mathbf{v}}{dt}\right)_T + \mathbf{N}\left(\frac{d\mathbf{v}}{dt}\right)_N,$$

that is, the acceleration vector is always located in the osculating plane of the trajectory (see, for instance, C. Mataix Aracil, Cálculo Vectorial Intrí nseco, Third Edition, Madrid 1951, page 49). Combined with the equation for v, this last equation produces

$$v = \frac{1}{2}\frac{\mathbf{F} \cdot (\mathbf{T} \times \mathbf{N})}{\boldsymbol{\omega} \cdot (\mathbf{T} \times (\mathbf{T} \times \mathbf{N}))} = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{B})}{(\boldsymbol{\omega} \cdot \mathbf{N})} = \frac{1}{2}\frac{F_B}{\omega_N}.$$

The scalar factor $(d\mathbf{v}/dt)_N$ can be omitted. We therefore obtain an equation that does not explicitly contain the acceleration, but that besides $\mathbf{F}$ and $\boldsymbol{\omega}$, only contains purely geometric quantities.

Taking into consideration the relationships between the unit vectors $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{N}$, we easily recognize that the following equations are also valid:

$$v = \frac{1}{2}\frac{(\mathbf{F} \cdot (\mathbf{N} \times \mathbf{T}))}{(\boldsymbol{\omega} \cdot \mathbf{N})},$$

$$v = \frac{1}{2}\frac{(\mathbf{F} \cdot \mathbf{B})}{(\boldsymbol{\omega} \cdot (\mathbf{T} \times \mathbf{B}))}.$$

Each equation in these two cases contains two of the three independent parameters; $\mathbf{T}$ determines the normal plane. For instance, if we fix the normal plane, that is, the vector $\mathbf{T}$ through its two independent components, we will still need a direction in this normal plane to determine $\mathbf{B}$ or $\mathbf{N}$.

5. Simpler solution

We'll finish with a second demonstration:

If we now write Euler's equation of motion as follows:

$$\mathbf{T}\left(\frac{d\mathbf{v}}{dt}\right)_T + \mathbf{N}\left(\frac{d\mathbf{v}}{dt}\right)_N + 2v(\boldsymbol{\omega} \times \mathbf{T}) = \mathbf{F}$$

and take the scalar product with the binormal unit vector $\mathbf{B}$, we find

$$2v\,(\boldsymbol{\omega} \times \mathbf{T}) \cdot \mathbf{B} = 2v\,\boldsymbol{\omega} \cdot (\mathbf{T} \times \mathbf{B}) = 2v\,(\boldsymbol{\omega} \cdot \mathbf{N}) = \mathbf{F} \cdot \mathbf{B}.$$

As seen above, this equation of motion in the binormal unit direction of the moving trihedron yields also the relationship between the wind velocity and the intensity of the atmospheric hydrodynamic field.

10 ERTEL (1960a): Theorem on hydrodynamical material invariants

Summary

The following theorem is easily demonstrated: The product of the specific volume by the functional determinant of three material invariants with respect to orthogonal cartesian spatial coordinates is also a material invariant.

1. Symbols

We use tensor notation and omit the summation symbol for indices that appear twice. We define the following symbols: x_i $(i = 1, 2, 3)$ = rectangular cartesian coordinates, v_i $(i = 1, 2, 3)$ = components of the fluid velocity, δ_{jk} $(j, k = 1, 2, 3)$ = substitution tensor, $(\cdots)_{,j}$ = partial derivative of $(\cdots)$ with respect to x_j, $d(\cdots)/dt = \partial(\cdots)/\partial t + v_j(\cdots)_{,j}$ = material derivative of $(\cdots)$, $\sigma = 1/\rho$ = specific volume (ρ = density) of the fluid, and t = time.

2. Concept of a material invariant

Let $f(x_1, x_2, x_3, t)$ be a property of a fluid particle; then

$$\frac{df}{dt} = \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} \right) f + \frac{\partial f}{\partial t} \quad (10.1)$$

is called a material derivative because df/dt measures the variation of f as a function of time following the fluid particle. If $F(x_1, x_2, x_3, t)$ is conserved following the parcel, then $dF/dt = 0$ and therefore we can write

$$\left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} \right) F = -\frac{\partial F}{\partial t},$$

or simply

$$v_j F_{,j} = -\frac{\partial F}{\partial t}, \quad (10.2)$$

using tensor notation.

In this case, F is conserved along the trajectory of a moving particle. For this reason F is also called a material constant. There are many applications for a material invariant in meteorology and oceanography.

3. A theorem on material invariants

Let J_k $(k = 1, 2, 3)$ be three material invariants. The Jacobian (k = subindex of rows, j = subindex of columns)

$$J = |J_{k,j}| = \frac{\partial(J_1, J_2, J_3)}{\partial(x_1, x_2, x_3)} \quad (10.3)$$

is not equal to zero, that is, J_1, J_2, and J_3 are independent.

Moreover, it is necessary that the following three conditions are met:

$$v_j J_{k,j} = -\frac{\partial J_k}{\partial t}. \quad (10.4)$$

Denote the cofactor of $J_{k,i}$ in J as $\overset{*}{J}_{ki}$. Using this notation we have

$$\overset{*}{J}_{ki} J_{k,j} = \delta_{ij} J = \begin{cases} J, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (10.5)$$

and as solutions of the system of equations in (10.4), with v_i $(i = 1, 2, 3)$ as unknowns and using (10.5), we obtain:

$$J v_i = -\frac{\partial J_k}{\partial t} \overset{*}{J}_{ki}. \quad (10.6)$$

The divergence of the vector $J v_i$ is then given by

$$(J v_i)_{,i} = -\frac{\partial J}{\partial t}, \quad (10.7)$$

because

$$\left(\overset{*}{J}_{ki} \right)_{,i} = \overset{*}{J}_{ki,i} \equiv 0 \quad (10.8)$$

for all k; moreover

$$\frac{\partial J_{k,i}}{\partial t} \overset{*}{J}_{ki} = \frac{\partial J}{\partial t} \quad (10.9)$$

is the derivative of the determinant J with respect to t.

From the continuity equation

$$\frac{d\sigma}{dt} = \sigma v_{i,i} \quad (10.10)$$

equation (10.7) can be written as

$$\sigma \left(\frac{\partial J}{\partial t} + v_i J_{,i} \right) + J \frac{d\sigma}{dt} = 0, \quad (10.11)$$

which can be rewritten as:

$$\boxed{\frac{d}{dt}(\sigma J) = \frac{d}{dt} \left\{ \sigma \frac{\partial(J_1, J_2, J_3)}{\partial(x_1, x_2, x_3)} \right\} = 0.} \quad (10.12)$$

Therefore, the following theorem is verified:

The product of the specific volume by the functional determinant of three material invariants with respect to orthogonal cartesian spatial coordinates is also a material invariant.

Analogous results are obtained in the general case of n material derivatives $J_k(x_1, x_2, x_3, ..., x_n, t)$ $(k = 1, 2, 3, ..., n)$ with n orthogonal cartesian coordinates of an n-dimensional space, assuming that $J_1, J_2, J_3, \ldots, J_n$ are independent.

4. Meteorological examples

Isentropic motion in vertical columns of the atmosphere is an example of an application of equation (10.7). Assume an isentropic air motion in the vertical direction, $x_3 = z$. All particles move parallel to the vertical axis without lateral expansion or contraction and without change of specific entropy, S, so that there are three material invariants

$$\left.\begin{array}{l} J_1 = x_1 = x, \\ J_2 = x_2 = y, \\ J_3 = S(x, y, z, t). \end{array}\right\} \quad (10.13)$$

These functions are independent in the region of study. The functional determinant (10.3) can be written as

$$J = \frac{\partial(x, y, S)}{\partial(x, y, z)} = \frac{\partial S}{\partial z}, \quad (10.14)$$

and equation (10.12) gives the well-known result

$$\frac{d}{dt}\left(\sigma \frac{\partial S}{\partial z}\right) = 0. \quad (10.15)$$

A so-called Margules material invariant, $\sigma(\partial S/\partial z)$, appears in this equation.

The following equality is sometimes convenient

$$J = \frac{\partial(J_1, J_2, J_3)}{\partial(x_1, x_2, x_3)} = (\nabla J_1 \times \nabla J_2) \cdot \nabla J_3 \quad (10.16)$$

(see, for example, ERTEL [1], MORÁN SAMANIEGO [2]).

References

[1] H. ERTEL, H., Winddivergenz auf Isobarenflächen und Luftdruckänderung. Met. Z. *60* (1943), 188–191.

[2] MORÁN SAMANIEGO, F., Apuntes de Termodinámica de la Atmósfera, Madrid 1944, 70–81.

11 ERTEL (1960b): Relation between a material derivative and a given spatial divergence in hydrodynamics

Summary

We derive a transformation that shows how the material derivative of any function can be expressed as a spatial divergence multiplied by the fluid's specific volume. This relationship shows, for instance, the different forms of the continuity equation (Euler, Lagrange) as special cases.

1. Symbols, notation, and fundamental concepts

We define the following symbols: x_i $(i = 1, 2, 3) = $ rectangular cartesian coordinates, v_i $(i = 1, 2, 3) = $ components of the fluid velocity, $t = $ time, $\sigma = $ specific volume, $\rho = $ density $(= 1/\sigma)$, F_i $(i = 1, 2, 3) = $ components of the external force, p_{ij} $(i, j = 1, 2, 3) = $ pressure tensor, and δ_{ij} $(i, j = 1, 2, 3) = $ substitution tensor.

In order to simplify the notation we use the summation rule of tensorial calculus in which the use of index repetition indicates summation. Indices that are not repeated will take on values of 1,2,3 successively so that there is a total of three different equations.

It is convenient to write $\psi_{,j}$ instead of $\partial\psi/\partial x_j$. With this notation $\Theta_{k,j} = \partial\Theta_k/\partial x_j$, $p_{ij,j} = \partial p_{ij}/\partial x_j$, etc. (see, for instance, VOGEL [4]).

Let $\psi = \psi(x_1, x_2, x_3, t)$ be a differentiable function whose physical meaning is the component of a tensor of any order. The operator

$$\frac{\partial\psi}{\partial t} + v_j\psi_{,j} = \frac{d\psi}{dt} \quad (11.1)$$

is the so-called material derivative of ψ with respect to t. We know that (11.1) represents a derivative following the motion of a particle which at time t is located at x_j and has velocity $dx_j/dt = v_j$ $(j = 1, 2, 3)$.

Introducing the continuity equation

$$\frac{d\sigma}{dt} = \sigma v_{j,j} \quad (11.2)$$

we can write the material derivative in (11.1) as

$$\frac{d\psi}{dt} = \sigma\left\{(\rho v_j\psi)_{,j} + \frac{\partial(\rho\psi)}{\partial t}\right\} \quad (11.3)$$

We see that the material derivative is here expressed as a four-dimensional divergence (a spatio-temporal divergence) multiplied by the fluid's specific volume. Equation (11.3) is often found in conservation principles of theoretical physics (see, for instance, SCHRÖDINGER [3]).

Euler's fundamental hydrodynamical equations contain both material derivatives and spatial divergences:

$$\frac{dv_i}{dt} + \sigma\, p_{ij,j} = F_i. \qquad (11.4)$$

The continuity equation (11.2) also has this property. It is therefore convenient to express all material derivatives as spatial divergences. In the next section, we will show that this transformation is possible with the help of functional determinants (Jacobian and Hessian).

2. Transformation of the material derivative to a spatial divergence

After the appropriate choice of three functions

$$\Theta_k = \Theta_k(x_1, x_2, x_3, t) \qquad (k = 1, 2, 3)$$

it is possible to express a function $\psi = \psi(x_1, x_2, x_3, t)$ as the Jacobian $J = J(\Theta_1, \Theta_2, \Theta_3)$ multiplied by the specific volume, as follows:

$$\psi = \sigma\, \frac{\partial(\Theta_1, \Theta_2, \Theta_3)}{\partial(x_1, x_2, x_3)} = \sigma\, |\Theta_{k,j}| = \sigma J \qquad (11.5)$$

(k = subindex of rows, j = subindex of columns). This kind of representation is always possible when $\Theta_1, \Theta_2, \Theta_3$ are independent, that is:

$$J = |\Theta_{k,j}| \neq 0. \qquad (11.6)$$

If we designate as $\overset{*}{\Theta}_{kj}$ the adjoint determinant of $\Theta_{k,j}$ in $J = |\Theta_{k,j}|$, a well-known property of the determinants tells us that

$$\overset{*}{\Theta}_{ki}\Theta_{k,j} = \delta_{ij}J. \qquad (11.7)$$

Consider now the identities

$$\frac{\partial \Theta_k}{\partial t} + v_j \Theta_{k,j} = \frac{d\Theta_k}{dt}. \qquad (11.8)$$

If we multiply these equations by $\overset{*}{\Theta}_{ki}$, and sum over all k, use of (11.7) gives

$$\frac{\partial \Theta_k}{\partial t}\overset{*}{\Theta}_{ki} + v_i J = \overset{*}{\Theta}_{ki}\frac{d\Theta_k}{dt}. \qquad (11.9)$$

From (11.9) we'd like to obtain the spatial divergence; we then have

$$\frac{\partial J}{\partial t} + (v_i J)_{,i} = \left(\overset{*}{\Theta}_{ki}\frac{d\Theta_k}{dt}\right)_{,i} \qquad (11.10)$$

because for every k

$$\left(\overset{*}{\Theta}_{ki}\right)_{,i} = \overset{*}{\Theta}_{ki,i} = 0, \qquad (11.11)$$

while

$$\frac{\partial \Theta_{k,i}}{\partial t}\overset{*}{\Theta}_{ki} = \frac{\partial J}{\partial t} \qquad (11.12)$$

is nothing but the partial derivative of the functional determinant J with respect to t.

Considering the continuity equation (11.2) we then have, according to (11.10):

$$\sigma\left(\frac{\partial J}{\partial t} + v_i J_{,i}\right) + J\frac{d\sigma}{dt} = \sigma\left(\Theta_{kj}\frac{d\Theta_k}{dt}\right)_{,j} \qquad (11.13)$$

and taking equation (11.1) into account, we obtain the needed transformation:

$$\boxed{\frac{d}{dt}\left\{\sigma\,\frac{\partial(\Theta_1, \Theta_2, \Theta_3)}{\partial(x_1, x_2, x_3)}\right\} = \sigma\left(\overset{*}{\Theta}_{kj}\frac{d\Theta_k}{dt}\right)_{,j}}$$
$$(11.14)$$

or

$$\frac{d\psi}{dt} = \sigma\left(\overset{*}{\Theta}_{kj}\frac{d\Theta_k}{dt}\right)_{,j} \qquad (11.15)$$

if we remember that ψ is given by (11.5).

Through this relationship it is possible to transform the material derivative of a function $\psi = \sigma J(\Theta_1, \Theta_2, \Theta_3)$ into a spatial divergence. This transformation is also valid for a Euclidian space of any dimension.

It is important to point out that in (11.15) the set of functions Θ_k ($k = 1, 2, 3$) contains two arbitrary functions, providing ∞^2 ways of representation if we express ψ/σ using the Jacobian in (11.5). If we assume that ψ/σ, Θ_1 and Θ_2 are given, then equation (11.5) determines Θ_3 through a first order partial differential equation.

For clarity let us consider the function $\psi = \sigma$, and $\Theta_1 = x_1$, $\Theta_2 = x_2$. Then, equation (11.5) requires that $\Theta_3 = x_3$. Therefore, $\overset{*}{\Theta}_{kj} = \delta_{jk}$, and equation (11.14) yields

$$\frac{d\sigma}{dt} = \sigma\left(\delta_{jk}\frac{dx_k}{dt}\right)_{,j} = \sigma\, v_{j,j} \qquad (11.16)$$

that is, the transformation in (11.15) yields Euler's continuity equation.

If we assume that $d\Theta_k/dt$ is a function of Θ_1, Θ_2 and Θ_3, we can establish that

$$\left(\overset{*}{\Theta}_{kj}\frac{d\Theta_k}{dt}\right)_{,j} = J\frac{\partial}{\partial \Theta_j}\left(\frac{d\Theta_j}{dt}\right)$$

33

in virtue of

$$\overset{*}{\Theta}_{kj}\left(\frac{d\Theta_k}{dt}\right)_{,j} = \overset{*}{\Theta}_{kj}\Theta_{i,j}\frac{\partial}{\partial\Theta_i}\left(\frac{d\Theta_k}{dt}\right)$$

$$= \delta_{ik}J\frac{\partial}{\partial\Theta_i}\left(\frac{d\Theta_k}{dt}\right) = J\frac{\partial}{\partial\Theta_j}\left(\frac{d\Theta_j}{dt}\right),$$

so that equation (11.14) becomes the equivalent form:

$$\boxed{\frac{d}{dt}\ln\left\{\sigma\frac{\partial(\Theta_1,\Theta_2,\Theta_3)}{\partial(x_1,x_2,x_3)}\right\} = \frac{\partial}{\partial\Theta_j}\left(\frac{d\Theta_j}{dt}\right),}$$

(11.14a)

which is advantageous when two functions (Θ_1 and Θ_2, for instance) from the set Θ_j ($j = 1,2,3$) are canonically conjugate variables. (For further information on canonically conjugate variables in hydrodynamics see LAMB [2]). If, moreover, Θ_3 is a material invariant, that is $d\Theta_3/dt = 0$, we have

$$\frac{\partial}{\partial\Theta_j}\left(\frac{d\Theta_j}{dt}\right) = 0,$$

(11.17)

and therefore, from (11.14a):

$$\frac{d}{dt}\left\{\sigma\frac{\partial(\Theta_1,\Theta_2,\Theta_3)}{\partial(x_1,x_2,x_3)}\right\} = 0$$

(11.18)

that is

$$\sigma\frac{\partial(\Theta_1,\Theta_2,\Theta_3)}{\partial(x_1,x_2,x_3)} = \text{material invariant.}$$

(11.19)

Our transformation also includes the following theorem: If the three independent functions Θ_1, Θ_2, and Θ_3 are individual invariants:

$$\frac{d\Theta_1}{dt} = \frac{d\Theta_2}{dt} = \frac{d\Theta_3}{dt} = 0$$

(11.20)

the product of the specific volume by the functional determinant of Θ_1, Θ_2 and Θ_3 with respect to the rectangular cartesian coordinates is also an individual invariant (see ERTEL [1]).

The initial position coordinates $\overset{\circ}{x}_1 = a_1 = \Theta_1$, $\overset{\circ}{x}_2 = a_2 = \Theta_2$, $\overset{\circ}{x}_3 = a_3 = \Theta_3$ at a reference time $t = 0$ are material invariants:

$$\left.\begin{array}{l}\dfrac{d\Theta_1}{dt} = \dfrac{da_1}{dt} = 0,\\[2mm]\dfrac{d\Theta_2}{dt} = \dfrac{da_2}{dt} = 0,\\[2mm]\dfrac{d\Theta_3}{dt} = \dfrac{da_3}{dt} = 0.\end{array}\right\}$$

(11.21)

Using (11.21), equation (11.14) becomes

$$\frac{d}{dt}\left\{\sigma\frac{\partial(a_1,a_2,a_3)}{\partial(x_1,x_2,x_3)}\right\} = 0,$$

(11.22)

which is a well-known equality known as Lagrange's continuity equation.

Finally, replacing the three functions Θ_k ($k = 1,2,3$) with a function Φ, which has the property that $\Phi_k = \Phi_{,k}$ ($k = 1,2,3$), turns the Jacobian (11.6) into the Hessian

$$H(\Phi) = |\Phi_{,kj}| = |\Phi_{,jk}|.$$

(11.23)

If we designate $\overset{*}{\Phi}_{kj}$ as the cofactor (minor with sign) of $\Phi_{,kj}$ in $H(\Phi)$, we have

$$\Phi_{,ki}\overset{*}{\Phi}_{kj} = \delta_{ij}H(\Phi)$$

(11.24)

instead of (11.7). Therefore, written as

$$\boxed{\frac{d}{dt}\{\sigma H(\Phi)\} = \sigma\left(\overset{*}{\Phi}_{kj}\frac{d\Phi_{,k}}{dt}\right)_{,j}}$$

(11.25)

we have a special case of equation (11.14). Using (11.25) we can transform the material derivative of a function ψ into a spatial divergence, when ψ is determined by a function Φ in the Hessian $H(\Phi) = \psi/\sigma$; however, in order to determine Φ we now have a nonlinear partial differential equation. It is immediately seen that (11.14), (11.14a) and (11.25) are of the form of the continuity equation.

References

[1] ERTEL, H., Teorema sobre invariantes sustanciales de la Hidrodinámica. Gerl. Beitr. Geophys. *69* (1960), 290–293.

[2] LAMB, H., Hydrodynamics. First American edition, New York 1945, 248–249.

[3] SCHRÖDINGER, E., Space-Time Structure, Cambridge 1950, 87–90.

[4] VOGEL, TH., Physique mathématique classique, Paris 1956, 40–43.

12 ERTEL (1961): Isogons and isotachs in two-dimensional potential flow of an incompressible fluid

Summary

Let $\theta = \text{const}$ and $v = \text{const}$ be the isogons and isotachs that correspond to two-dimensional, irrotational, and incompressible flow. Without making use of functional theoretical methods, the following theorem can be demonstrated: a rotation from $\nabla\theta$ to ∇v along the shortest path must always be a counterclockwise rotation of magnitude $\pi/2$, if θ is measured positive in the same direction.

1. Introduction

Many geophysical problems can be studied within the framework of two-dimensional, irrotational, and incompressible fluid motion. This is the kind of problem addressed here. Therefore, the description of a two-dimensional coordinate system suffices.

The hydrodynamical equations are reduced to:

$$\frac{\partial v_x}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \right) v_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad (12.1)$$

$$\frac{\partial v_y}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \right) v_y = -\frac{\partial \Phi}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad (12.2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad (12.3)$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0, \quad (12.4)$$

where x, y are rectangular cartesian coordinates, v_x, v_y the components of velocity, Φ the external force potential, p the pressure, and ρ the density.

In equations (12.1) and (12.2) the convective portions of the acceleration are

$$B_x = v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial x}\left(\frac{v^2}{2} \right), \quad (12.5)$$

$$B_y = v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{v^2}{2} \right), \quad (12.6)$$

where $v = |\mathbf{v}|$ is the absolute value of the vector $\mathbf{v} = (v_x, v_y)$.

Based on conditions (12.3) and (12.4) we will investigate the isogons and isotachs of the given hydrodynamical field.

2. Isogons and isotachs

Let x, y be a right-handed coordinate system. Using equation (12.3) we replace $\partial v_x/\partial x$ with $-\partial v_y/\partial y$ in (12.5) and $\partial v_y/\partial y$ with $-\partial v_x/\partial x$ in (12.6) and, after some manipulation, obtain (see, for instance, [1]):

$$\left.\begin{aligned} B_x = -v^2 \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial x}\left(\frac{v^2}{2} \right), \\ B_y = +v^2 \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial y}\left(\frac{v^2}{2} \right), \end{aligned}\right\} \quad (12.7)$$

where

$$v_x = v\cos\theta, \qquad v_y = v\sin\theta, \quad (12.8)$$

implies that the angle

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x} \right) \quad (12.9)$$

is measured in a counterclockwise manner. The equation $\theta(x, y, t) = \text{const}$ represents an isogon and the equation $v(x, y, t) = \text{const}$ represents an isotach.

We deduce from (12.7) that

$$\frac{\partial \theta}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial v}{\partial y} = 0, \quad (12.10)$$

which is an orthogonality condition that states that isogons and isotachs are perpendicular at all points where they intersect (KOZENY [2]).

From equation (12.7) we can derive the Cauchy-Riemann conditions:

$$\frac{\partial \ln v}{\partial x} = -\frac{\partial \theta}{\partial y}, \qquad \frac{\partial \ln v}{\partial y} = \frac{\partial \theta}{\partial x}; \quad (12.11)$$

on the other hand

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 2v \left(\frac{\partial \theta}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial \theta}{\partial y}\frac{\partial v}{\partial x} \right)$$
$$= 2v|\nabla\theta||\nabla v|\sin\psi, \quad (12.12)$$

where ψ is the angle formed by $\nabla\theta$ and ∇v. The angle ψ is also measured counterclockwise.

Consider now the expression for the divergence $\partial B_x/\partial x + \partial B_y/\partial y$ as a function of the components of the deformation tensor. Differentiating equations (12.5) and (12.6) and taking into account condition (12.3), we obtain:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \left(\frac{\partial v_x}{\partial x} \right)^2 + 2\frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x} + \left(\frac{\partial v_y}{\partial y} \right)^2. \quad (12.13)$$

Recalling that in our case the components of the deformation tensor can be reduced, using (12.4), to

$$\left.\begin{aligned} D_{xx} &= \frac{\partial v_x}{\partial x}, \\ D_{xy} = D_{yx} &= \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}, \\ D_{yy} &= \frac{\partial v_y}{\partial y}, \end{aligned}\right\} \quad (12.14)$$

we obtain the formula:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = D_{xx}^2 + 2D_{xy}^2 + D_{yy}^2. \qquad (12.15)$$

Comparing equations (12.12) and (12.15) we obtain

$$2v|\nabla\theta||\nabla v|\sin\psi = D_{xx}^2 + 2D_{xy}^2 + D_{yy}^2. \qquad (12.16)$$

It is easily seen that the condition for the first member of this equation is

$$\sin\psi > 0, \qquad (12.17)$$

since it is necessary to exclude the exceptional case when D_{xx}, D_{xy}, and D_{yy} simultaneously vanish at each point of the field, which describes the movement of a rigid body without rotation.

The orthogonality condition (12.10) of the vectors $\nabla\theta$ and ∇v determines two possible values for ψ:

$$\psi = \pm\frac{\pi}{2}, \qquad (12.18)$$

but the inequality in equation (12.17) can only be satisfied if

$$0 < \psi < \pi, \qquad (12.19)$$

a condition that excludes the possibility of a clockwise rotation. We immediately obtain the single solution

$$\boxed{\psi = +\frac{\pi}{2},} \qquad (12.20)$$

which satisfies the conditions established in (12.10) and (12.19), that is: the smallest rotation that goes from $\nabla\theta$ to ∇v is always a $\pi/2$ angle, if θ is measured in the same direction (see also KOZENY, loc. cit.).

Considering how we obtained conditions (12.10) and (12.19), it is easy to see that this theorem is valid for both stationary and non-stationary motion, because this result is simply a consequence of the kinematic conditions (12.3) and (12.4). The flow must, however, be two-dimensional and irrotational for an incompressible fluid.

References

[1] ERTEL, H., Orthogonale Trajektoriensysteme in stationären ebenen Strömungsfeldern inkompressibler idealer Flüssigkeiten. In: Deutsche Akademie der Wissenschaften zu Berlin 1946–1956, Berlin 1956, 67–91.

[2] KOZENY, J., Hydraulik, Wien 1953, 292.

13 ERTEL (1963a): Analogy between the equations of motion and the vortex equations in hydrodynamics

Summary

An analogy between Euler's motion equations and Helmholtz's vortex equations can be found through two simple transformations.

I. Notation and symbols

Tensorial notation is used. All indices vary between 1 and 3; summations are made over the repeated index. ϵ_{ijk} is the alternating tensor (see, for example, F. MORÁN [4]). x_i are the rectangular cartesian coordinates, $\mathbf{v} = \{v_i\}$ is the velocity vector ($|\mathbf{v}| = v$), $\boldsymbol{\xi} = \{\xi_i\}$ the vorticity vector, Φ the external force potential, p the pressure, ρ the density, and t the time.

II. The problem

We are accustomed to writing Euler's fundamental equations of hydrodynamics as

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\Phi + \frac{p}{\rho}\right), \qquad (13.1)$$

$$\frac{\partial v_j}{\partial x_j} = 0, \qquad (13.2)$$

for a homogeneous and incompressible fluid. Taking the curl of (13.1), using (13.2) and substituting the expression for vorticity which is given by

$$\xi_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}, \qquad (13.3)$$

we obtain Helmholtz's famous vorticity equations

$$\frac{\partial \xi_i}{\partial t} + v_j \frac{\partial \xi_i}{\partial x_j} = \xi_j \frac{\partial v_i}{\partial x_j}, \qquad (13.4)$$

(see, for example, M. LUCINI [3]).

Understandably, the equations of motion (13.1) do not coincide with the equations of vorticity (13.4). However, it is possible to show that there is an analogy between equations (13.1) and (13.4). In order to achieve

this analogy it is necessary to transform both systems (13.1) and (13.4). The demonstration follows.

III. Representation of the analogy

Adding the identities

$$v_j \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{v^2}{2} \right) \qquad (13.5)$$

to equations (13.1) it is possible to express the motion equations in the following form:

$$\boxed{\frac{\partial v_i}{\partial t} + v_j \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) = \frac{\partial L}{\partial x_i},} \qquad (13.6)$$

where

$$L = \frac{v^2}{2} - \left(\Phi + \frac{p}{\rho} \right) \qquad (13.7)$$

is Lagrange's function. The symmetric tensor

$$D_{ij}(\mathbf{v}) = \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \qquad (13.8)$$

measures the deformation of the velocity field.

Let us now consider the vorticity equations (13.4). Using the identities (see H. ERTEL [1]):

$$\xi_j \frac{\partial v_i}{\partial x_i} = \xi_j \frac{\partial v_j}{\partial x_i} = \frac{\partial (\xi_j v_j)}{\partial x_i} - v_j \frac{\partial \xi_j}{\partial x_i}, \qquad (13.9)$$

one can immediately write the vorticity equations (13.4) as follows:

$$\boxed{\frac{\partial \xi_i}{\partial t} + v_j \left(\frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j} \right) = \frac{\partial M}{\partial x_i},} \qquad (13.10)$$

where M is the scalar function (see, for example, H. ERTEL and C.-G. ROSSBY [2], K. OSWATITSCH [5])

$$M = \xi_j v_j. \qquad (13.11)$$

The symmetric tensor

$$D_{ij}(\boldsymbol{\xi}) = \frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j} \qquad (13.12)$$

measures the deformation of the vorticity field.

The equations we just derived, namely equations (13.6) and (13.10), express the analogy we were searching for. In this analogy there is a perfect correspondence between the partial derivatives

$$\frac{\partial v_i}{\partial t}, \frac{\partial v_j}{\partial x_i}, \frac{\partial v_i}{\partial x_j}, \frac{\partial L}{\partial x_i},$$

which appear in the motion equations (13.6), and

$$\frac{\partial \xi_i}{\partial t}, \frac{\partial \xi_j}{\partial x_i}, \frac{\partial \xi_i}{\partial x_j}, \frac{\partial M}{\partial x_i},$$

which appear in the vorticity equations (13.10).

One can also obtain generalized equations for compressible fluids, if we assume that the density $\rho = \rho(p)$, as a function of pressure, contains identical parameters for all particles. In this case the analogy sought is obtained by substituting p/ρ by $\int dp/\rho$, and ξ_i and ξ_j by ξ_i/ρ and ξ_j/ρ, respectively, in the corresponding expressions.

References

[1] ERTEL, H., Ein System von Identitäten und seine Anwendung zur Transformation von Wirbelgleichungen der Hydrodynamik. Mber. Dt. Akad. Wiss. Berlin *4* (1962), 292.

[2] ERTEL, H., and C.-G. ROSSBY, A new Conservation-Theorem of Hydrodynamics. Geofis. pura e appl. *XIV* (1949), Fasc. 3–4.

[3] LUCINI, M., Teoria de la Mecánica y sus aplicaciones, Tercera edición, Barcelona 1959, 510 y 653.

[4] MORÁN, F., Los tensores cartesianos rectangulares, Madrid 1954, 8.

[5] OSWATITSCH, K., Physikalische Grundlagon der Strömungslehre. Handb. d. Physik, Bd. VIII/1, Berline/Göttingen/Heidelberg 1959, 38.

14 ERTEL (1963b): Relationships between differential operators from vector calculus and Lagrange's bracket, with hydrodynamic applications

Summary

We will demonstrate the following theorem as a hydro-dynamical application of a more general problem: The curl of the material derivative of the vorticity of rotating particles in an ideal and homogeneous fluid may be represented using Lagrange's bracket.

For the sake of brevity we make use of tensor notation. All indices go from 1 to 3 and index repetition indicates summation. Later on, we will see that it is useful to introduce the permutation tensor ϵ_{ijk} and the Kronecker δ_{ij} [see, for instance, LAWDEN (^{2}), F. MORÁN (^{3}), TH. VOGEL (^{5})].

We choose a rectangular cartesian coordinate system, with axes x_j and introduce a system with six functions R_m and S_m of the x_j and t (=time) to form the symbols

$$[i,j] = \frac{\partial R_m}{\partial x_i}\frac{\partial S_m}{\partial x_j} - \frac{\partial R_m}{\partial x_j}\frac{\partial S_m}{\partial x_i} = -[j,i]. \quad (14.1)$$

These so-called Lagrange brackets are very important tools in Analytical Mechanics [see, for instance, D. MORGENSTERN and I. SZABÓ (^{4}), E. T. WHITTAKER (^{6})].

Suppose now that the system of six equations R_m and S_m is chosen such that the additional conditions

$$(S_1, S_2, S_3) = \mathbf{S} = \text{solenoidal vector} \quad (14.2)$$

and

$$(R_1, R_2, R_3) = \mathbf{R} = \text{rot}\,\mathbf{S} \quad (14.3)$$

are met.

In this case (14.1) becomes

$$\boxed{[i,j] = \epsilon_{ijk}\,\text{rot}_k\{(T\,\text{rot} - \text{rot}\,T)\mathbf{S}\}} \quad (14.4)$$

where T is the scalar operator

$$T = \mathbf{S} \cdot \nabla = S_m \frac{\partial}{\partial x_m}. \quad (14.5)$$

The demonstration of (14.4) is easily done as follows. From the definition of operator T, applied to $\mathbf{S}$, we obtain

$$T\mathbf{S} = \nabla\left\{\tfrac{1}{2}(\mathbf{S})^2\right\} - \mathbf{S} \times \mathbf{R}. \quad (14.6)$$

Now, following a simple transformation from vectorial calculus, we obtain

$$\text{rot}\,(\mathbf{S} \times \mathbf{R}) = (\mathbf{R} \cdot \nabla)\mathbf{S} - T\mathbf{R} \quad (14.7)$$

as the divergence of a curl vanishes, and also, due to condition (14.2), $\text{div}\,\mathbf{S} = 0$, so that

$$(\mathbf{R} \cdot \nabla)\mathbf{S} = (T\,\text{rot} - \text{rot}\,T)\mathbf{S}. \quad (14.8)$$

In order to obtain Lagrange's bracket, we take the curl of (14.8), whose component on the x_k axis is

$$\epsilon_{rsk}\frac{\partial}{\partial x_r}(\mathbf{R} \cdot \nabla S_s) \\ = \text{rot}_k\{(T\,\text{rot} - \text{rot}\,T)\mathbf{S}\}. \quad (14.9)$$

Moreover, we obtain

$$\mathbf{R} \cdot \nabla S_s = R_m\frac{\partial S_s}{\partial x_m} \quad (14.10)$$

and its equivalent

$$\mathbf{R} \cdot \nabla S_s = R_m\frac{\partial S_m}{\partial x_s}, \quad (14.11)$$

since the curl of $\mathbf{S}$ is defined as

$$\frac{\partial S_m}{\partial x_s} - \frac{\partial S_s}{\partial x_m} = \epsilon_{smn}\,\text{rot}_n\,\mathbf{S} = \epsilon_{smn}R_n, \quad (14.12)$$

and therefore

$$R_m\left(\frac{\partial S_m}{\partial x_s} - \frac{\partial S_s}{\partial x_m}\right) = \epsilon_{smn}R_mR_n \equiv 0, \quad (14.13)$$

[see also: H. ERTEL (^{1})].

Substituting the value of $\mathbf{R} \cdot \nabla S_s$, given by (14.11), into (14.9) yields

$$\epsilon_{rsk}\frac{\partial R_m}{\partial x_r}\frac{\partial S_m}{\partial x_s} = \text{rot}_k\{(T\,\text{rot} - \text{rot}\,T)\mathbf{S}\}. \quad (14.14)$$

Applying the permutation tensor ϵ_{ijk} to (14.14) and then using

$$\epsilon_{ijk}\,\epsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}, \quad (14.15)$$

we obtain

$$\frac{\partial R_m}{\partial x_i}\frac{\partial S_m}{\partial x_j} - \frac{\partial R_m}{\partial x_j}\frac{\partial S_m}{\partial x_i} \\ = \epsilon_{ijk}\,\text{rot}_k\{(T\,\text{rot} - \text{rot}\,T)\mathbf{S}\} \quad (14.16)$$

QED.

Let us now take a look at the rotating motion of a homogeneous, ideal liquid as a sample application of

(14.4). In this case there is an acceleration potential Ω; then

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Omega, \qquad (14.17)$$

where $\mathbf{v}$ is the velocity vector, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \qquad (14.18)$$

is the material derivative.

Keeping in mind the quantities we have previously mentioned, and since $\mathbf{v}$ is a solenoidal vector, we can establish the following relationships:

$$\mathbf{S} = \mathbf{v} \qquad (14.19)$$

$$\mathbf{R} = \nabla \times \mathbf{v} = \boldsymbol{\xi} = \text{vorticity}, \qquad (14.20)$$

$$T = \mathbf{v} \cdot \nabla = \frac{d}{dt} - \frac{\partial}{\partial t}. \qquad (14.21)$$

$$\qquad (14.22)$$

Moreover, in virtue of (14.17) and (14.21) we have

$$\nabla \times (T\mathbf{v}) = -\frac{\partial \boldsymbol{\xi}}{\partial t} \qquad (14.23)$$

$$(T\,\text{rot} - \text{rot}\,T)\mathbf{v} = \frac{d\boldsymbol{\xi}}{dt} \qquad (14.24)$$

Substituting this expression into equation (14.4), we have

$$\boxed{[i,j] = \epsilon_{ijk}\,\text{rot}_k\left(\frac{d\boldsymbol{\xi}}{dt}\right)} \qquad (14.25)$$

or, introducing the representation in terms of the components we have

$$\begin{cases} [2,3] = \text{rot}_1\left(\dfrac{d\boldsymbol{\xi}}{dt}\right), \\[2mm] [3,1] = \text{rot}_2\left(\dfrac{d\boldsymbol{\xi}}{dt}\right), \\[2mm] [1,2] = \text{rot}_3\left(\dfrac{d\boldsymbol{\xi}}{dt}\right), \end{cases} \qquad (14.26)$$

which demonstrates the theorem below.

The curl of the material derivative of the vorticity of rotating particles in a homogeneous and ideal fluid may be represented using Lagrange's brackets (non conservative).

This demonstration may also be done in a slightly different manner, by taking the appropriate transformation of the curl of Helmholtz's differential equation in vectorial notation. This, however, again requires use of the fundamental equalities in (14.10, 14.11).

References

(^{1}) H. ERTEL: *Ein System von Identitäten und seine Anwendung zur Transformation von Wirbelgleichungen der Hydrodynamik.* Monatsberichte der Dt. Akad. Wiss. Berlin. Bd. 4, 1962, 292.

(^{2}) D. F. LAWDEN: *An Introduction to Tensor Calculus and Relativity.* London-New York 1962, 23–37.

(^{3}) F. MORÁN: *Los tensores cartesianos rectangulares.* Madrid 1954, 1–9, 172.

(^{4}) D. MORGENSTERN and I. SZABÓ: *Vorlesungen über Theoretische Mechanik.* Berlin-Göttingen-Heidelberg 1961, 31, 40, 46.

(^{5}) TH. VOGEL: *Physique mathématique classique.* Paris 1956, 38.

(^{6}) E. T. WHITTAKER: *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Fourth Edition. Cambridge 1952, 298.

15 ERTEL (1964): The maximum number of independent invariants in hydrodynamics

Summary

A theorem from the theory of partial differential equations allows the demonstration that the maximun number of independent invariants in hydrodynamics is equal to the number of dimensions in the reference space.

In order to study the general characteristics of the movement in any fluid it is more convenient to use orthogonal coordinates $x_1, x_2 \ldots, x_n$ in a Euclidean space of n dimensions. If we project the successive positions of the fluid particles along their trajectories during a period of time t onto the coordinate axes, the n-dimensional velocity of the fluid particles will have the following n components

$$(v_1, v_2, \ldots, v_n) = \frac{d}{dt}(x_1, x_2, \ldots, x_n), \qquad (15.1)$$

where the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n} \quad (15.2)$$

indicates a derivative following the motion of the fluid.

Let $\psi = \psi(x_1, x_2 \cdots, x_n, t)$ be a point function that defines a property of the hydrodynamic field during a period of time t. If this property is stationary on the moving particles

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + v_1 \frac{\partial \psi}{\partial x_1} + v_2 \frac{\partial \psi}{\partial x_2} + \ldots + v_n \frac{\partial \psi}{\partial x_n} = 0 \quad (15.3)$$

which means that ψ remains constant if we move with one of the particles. When (15.3) is true, the variable ψ is called a 'material invariant' or 'hydrodynamic invariant' with respect to the trajectory considered. The study of these hydrodynamic invariants is very important in dynamic meteorology and magnetohydrodynamics (see, for example, [1], [2], [3], [4], [6], [8]).

The identification of a given particle in an n-dimensional fluid along its trajectory during a period of time $t \geq 0$ requires knowledge of n independent quantities which satisfy (15.3). As an example, we may use Lagrange's initial coordinates $\{a_1, a_2 \ldots, a_n\}$ which form a completely independent set.

All of the initial coordinates satisfy (15.3), and therefore the system of parametric equations for their trajectory

$$\left.\begin{array}{l} x_1 = x_1(a_1, a_2 \ldots, a_n, t), \\ x_2 = x_2(a_1, a_2 \ldots, a_n, t), \\ \vdots \\ x_n = x_n(a_1, a_2 \ldots, a_n, t), \end{array}\right\} \quad (15.4)$$

contains the n hydrodynamic invariants $a_1, a_2, \ldots, a_n$ as solutions of Lagrange's fundamental hydrodynamic equations.

By virtue of the continuity equation, we know that the determinant of the transformation cannot vanish:

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(a_1, a_2, \ldots, a_n)} \neq 0. \quad (15.5)$$

Therefore, solving (15.4) for $a_1, a_2, \ldots, a_n$, one obtains

$$\left.\begin{array}{l} a_1 = a_1(x_1, x_2, \ldots, x_n, t), \\ a_2 = a_2(x_1, x_2, \ldots, x_n, t), \\ \vdots \\ a_n = a_n(x_1, x_2, \ldots, x_n, t). \end{array}\right\} \quad (15.6)$$

Other hydrodynamic invariants can be added which correspond to certain thermodynamic conditions. An example of that is the entropy in adiabatic non-dissipative flows (see, for example, [9]).

One should, however, conjecture that more than n hydrodynamic invariants will be functionally dependent. The demonstration of this hypothesis is very simple: Given n invariants $a_1, a_2, \ldots, a_n$ and an additional different invariant J, the system of n conditions

$$\left.\begin{array}{l} \left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}\right) a_1 = 0, \\ \left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}\right) a_2 = 0, \\ \vdots \\ \left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}\right) a_n = 0, \end{array}\right\} \quad (15.7)$$

along with the additional condition

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}\right) J = 0 \quad (15.8)$$

constitutes a heterogeneous linear system of $n + 1$ equations with n unknowns $v_1, v_2, \ldots, v_n$ (where $J, a_1, a_2, \ldots, a_n$ and all of their first order partial derivatives are continuous functions). From algebraic analysis we know that a heterogeneous linear system of $n + 1$ equations and n unknowns will only admit non trivial solutions when the determinant formed by the coefficients and the independent terms vanishes (see, for example [7]).

The condition of compatibility of the system of equations comprised by (15.7) and (15.8) is

$$\begin{vmatrix} \frac{\partial J}{\partial t} & \frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2} & \cdots & \frac{\partial J}{\partial x_n} \\ \frac{\partial a_1}{\partial t} & \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial a_n}{\partial t} & \frac{\partial a_n}{\partial x_1} & \frac{\partial a_n}{\partial x_2} & \cdots & \frac{\partial a_n}{\partial x_n} \end{vmatrix} = \frac{\partial(J, a_1, a_2, \ldots, a_n)}{\partial(t, x_1, x_2, \ldots, x_n)}$$

$$= 0. \quad (15.9)$$

This condition implies functional dependence between the functions $J, a_1, a_2, \ldots, a_n$ of the variables $x_1, x_2, \ldots, x_n$:

$$\boxed{J = J(a_1, a_2, \ldots, a_n)} \quad (15.10)$$

(see, for example, [5]). It follows that *the maximum number of independent hydrodynamic invariants*

is equal to the number of dimensions of the reference space, as we wanted to prove.

This result may be enunciated as the application of a theorem from the theory of partial differential equations, namely: Any function J which obeys (15.8) should verify (15.9) (eliminating $v_1, v_2, \ldots, v_n$, combined with (15.7)), which proves that there is a functional dependence between $J, a_1, a_2, \ldots, a_n$ (see, for example, [10]).

References

[1] ERTEL, H., Über Hydrodynamische Wirbelätze. Physikal. Z. *43* (1942), 526–520.

[2] ERTEL, H., Teorema sobre invariantes sustanciales de la Hidrodinámica. Gerl. Beitr. Geophys. *69* (1960), 290–293.

[3] ERTEL, H., and C.-G. ROSSBY, A new Conservation-theorem of Hydrodynamics. Geofi s. pura y appl. *XIV* (1949), 189–193.

[4] FORTAK, H., Zur Frage allgemeiner hydrodynamischer Erhaltungssätze. Gerl. Beitr. Geophys. *65* (1956), 283–294.

[5] GILLESPIE, R.P., Partial Differentiation, Edinburgh/London 1951, 43–46.

[6] HOLLMANN, G., Ein vollständiges System hydrodynamischer Erhaltungssätze. Arch. Meterol. Geophys. Bioklim. A *14* (1964). 1–13.

[7] MATAIX ARACIL, C., Análisis algébrico e infi nitesimal, Tomo I, Quinta edicín Madrid 1957, 304–305.

[8] MAUERSBERGER, P., Zur Kinematik der magnetischen Feldlinien im Plasma. Pure appl. Geophys. *87* (1964), I, 143–148.

[9] MÓRAN SAMANIEGO, F., Apuntes de Termodinámica de la Atmósfera, Madrid 1944, 70–81.

[10] PUIG ADAM, P., Ecuaciones diferenciales aplicadas a la Física y Teénica, Madrid 1958, 214–215.

16 ERTEL (1965a): Hydrodynamic commutation formulas

Summary

From the Lagrangian form of the hydrodynamic equations for perfect fluids a commutation relation is derived. The continuity equation is not necessary for this derivation. This commutation relation, expressed in Lagrangian arguments, can be transformed by additional consideration of the continuity equation into a commutation relation expressed in Eulerian arguments.

1. Symbols and fundamental equations

In the following we will use the symbols of tensor analysis in three-dimensional Euclidean space. All indices range over the values 1,2,3. In a term with repeated indices we use the summation convention, with the sum taken from 1 to 3.

Define

$$\delta_{ij} = \begin{cases} +1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

as the substitution tensor and

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } i, j, k \text{ form a cyclic sequence}, \\ -1, & \text{if } i, j, k \text{ form a noncyclic sequence}, \\ 0, & \text{if any two indices agree}, \end{cases}$$

as the permutation symbol.

The hydrodynamic equations for a perfect gas can be written in the Lagrangian form

$$\frac{\partial^2 x_m}{\partial t^2} \frac{\partial x_m}{\partial a_j} = -\frac{\partial \Phi}{\partial a_j} - \sigma \frac{\partial p}{\partial a_j} \tag{16.1}$$

(momentum equation)

$$\sigma_0 = \sigma D^{-1} \tag{16.2}$$

(continuity equation)

or in the Eulerian form

$$\frac{dv_j}{dt} = \left(\frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} \right) v_j = -\frac{\partial \Phi}{\partial x_j} - \sigma \frac{\partial p}{\partial x_j} \tag{16.3}$$

(momentum equation)

$$\frac{d\sigma}{dt} = \sigma \frac{\partial v_k}{\partial x_k} \tag{16.4}$$

(continuity equation)

where we have defined the scalars Φ = potential for external forces, p = pressure, σ = specific volume (reciprocal of the density), t = time, and in (16.2) the functional determinant

$$D = \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} \tag{16.5}$$

of the position coordinates $x_j = x_j(a_1, a_2, a_3)$ of a fluid particle for $t \geq 0$ with respect to the labeling coordinates a_j, and the specific volume $\sigma_0 = \sigma_0(a_1, a_2, a_3)$ at $t = 0$.

In the Lagrangian framework (16.1) and (16.2) the independent variables are the labeling coordinates a_j and the time t, while in the Eulerian framework (16.3) and (16.4) the independent variables are the orthogonal cartesian coordinates x_j and the time t. The material derivative in the Lagrangian framework is denoted by the operator $\partial/\partial t$, while in the Eulerian framework by the operator $d/dt = \partial/\partial t + v_k(\partial/\partial x_k)$, so that, for example, the wind components in the Lagrangian framework are given by

$$\frac{\partial}{\partial t} x_j(a_1, a_2, a_3, t) = \frac{\partial x_j}{\partial t} = v_j \qquad (16.6)$$

and in the Eulerian framework by

$$\frac{dx_j}{dt} = \left(\frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} \right) x_j = v_k \frac{\partial x_j}{\partial x_k}$$
$$= v_k \delta_{jk} = v_j. \qquad (16.7)$$

The scalar fields p, σ, Φ in the Lagrangian framework are functions of the independent variables a_j, t and in the Eulerian framework are functions of the independent variables x_j, t.

2. Derivation of a commutation relation from the Lagrangian form of the hydrodynamical equations

We proceed from the Lagrangian momentum equations (16.1), which, in consideration of (16.6), can be written as

$$\frac{\partial v_m}{\partial t} \frac{\partial x_m}{\partial a_j} = -\frac{\partial \Phi}{\partial a_j} - \sigma \frac{\partial p}{\partial a_j}. \qquad (16.8)$$

We then apply to (16.8) the differential operator $\epsilon_{ijk} \frac{\partial \psi}{\partial a_k} \frac{\partial}{\partial a_i}$, containing the differentiable function ψ. Considering the antisymmetry of ϵ_{ijk}, we have:

$$\epsilon_{ijk} \frac{\partial \psi}{\partial a_k} \frac{\partial^2 v_m}{\partial t \, \partial a_i} \frac{\partial x_m}{\partial a_j} = -\epsilon_{ijk} \frac{\partial \sigma}{\partial a_i} \frac{\partial p}{\partial a_j} \frac{\partial \psi}{\partial a_k}$$
$$= +\epsilon_{ijk} \frac{\partial p}{\partial a_i} \frac{\partial \sigma}{\partial a_j} \frac{\partial \psi}{\partial a_k} \qquad (16.9)$$

or, again considering (16.6) and $\epsilon_{ijk} \frac{\partial v_m}{\partial a_i} \frac{\partial v_m}{\partial a_j} = 0$:

$$\frac{\partial}{\partial t} \left(\epsilon_{ijk} \frac{\partial v_m}{\partial a_i} \frac{\partial x_m}{\partial a_j} \frac{\partial \psi}{\partial a_k} \right)$$
$$- \left(\epsilon_{ijk} \frac{\partial v_m}{\partial a_i} \frac{\partial x_m}{\partial a_j} \frac{\partial}{\partial a_k} \right) \frac{\partial \psi}{\partial t}$$
$$= \epsilon_{ijk} \frac{\partial p}{\partial a_i} \frac{\partial \sigma}{\partial a_j} \frac{\partial \psi}{\partial a_k}. \qquad (16.10)$$

The right hand side of (16.10) is the functional determinant of the functions p, σ, ψ with respect to the coordinates a_1, a_2, a_3, i.e.,

$$\epsilon_{ijk} \frac{\partial p}{\partial a_i} \frac{\partial \sigma}{\partial a_j} \frac{\partial \psi}{\partial a_k} = \frac{\partial(p, \sigma, \psi)}{\partial(a_1, a_2, a_3)}. \qquad (16.11)$$

The two terms of the left side of (16.10) contain the Lagrangian bracket symbols from analytical mechanics (see, for example, E. T. WHITTAKER [16]), here in (16.10) for the quantities v_m and x_m ($m = 1, 2, 3$):

$$\epsilon_{ijk} \frac{\partial v_m}{\partial a_i} \frac{\partial x_m}{\partial a_j} = \{v, x\}_k \qquad (16.12)$$

where

$$\left. \begin{array}{l} v \equiv v_m(a_1, a_2, a_3, t) \\ \text{and} \\ x \equiv x_m(a_1, a_2, a_3, t) \end{array} \right\} \qquad (16.13)$$

are used as abbreviations for the wind components and the position coordinates.

The substitution of (16.12) into (16.10) results in the sought-after commutation relation in the form:

$$\boxed{ \begin{array}{l} \left(\frac{\partial}{\partial t} \right) \left(\{v, x\}_k \frac{\partial}{\partial a_k} \right) \psi - \left(\{v, x\}_k \frac{\partial}{\partial a_k} \right) \left(\frac{\partial}{\partial t} \right) \psi \\ \qquad = \epsilon_{ijk} \frac{\partial p}{\partial a_i} \frac{\partial \sigma}{\partial a_j} \frac{\partial \psi}{\partial a_k} \end{array} }$$
(16.14)

or, written in operator form:

$$\boxed{ \begin{array}{l} \left(\frac{\partial}{\partial t} \right) \left(\{v, x\}_k \frac{\partial}{\partial a_k} \right) - \left(\{v, x\}_k \frac{\partial}{\partial a_k} \right) \left(\frac{\partial}{\partial t} \right) \\ \qquad = \epsilon_{ijk} \frac{\partial p}{\partial a_i} \frac{\partial \sigma}{\partial a_j} \frac{\partial}{\partial a_k} \end{array} }$$
(16.15)

These commutation relations, (16.14) and (16.15), state that the operators

$$\frac{\partial}{\partial t} \qquad \text{and} \qquad \{v, x\}_k \frac{\partial}{\partial a_k} \qquad (16.16)$$

commute when applied to any (twice differentiable) function ψ if there exists for all fluid particles the same piezotropic relation

$$\sigma = \sigma(p). \qquad (16.17)$$

If the relationship (16.17) does not exist, the operators (16.16), when applied to a function of the special

42

form $\psi = \psi(p, \sigma)$, also commute, since in this case the functional determinant (16.11) likewise disappears. It is natural that the commutation property of the operators (16.16) always applies for homogeneous ($\sigma = $ const) perfect fluids, since constant σ makes the right sides of (16.14) and (16.15) vanish.

The commutation relations (16.14) and (16.15) make possible the derivation of conservation laws. Consider a fluid for which there exists one piezotropic relation of the form (16.17). We can select for ψ, for example, each labeling coordinate a_j. Since

$$\frac{\partial a_j}{\partial t} = 0 \qquad (16.18)$$

always holds, (16.14) reduces to

$$\left(\frac{\partial}{\partial t}\right)\left(\{v, x\}_k \frac{\partial a_j}{\partial a_k}\right) = \left(\frac{\partial}{\partial t}\right)(\{v, x\}_k \delta_{jk})$$
$$= \left(\frac{\partial}{\partial t}\right)(\{v, x\}_j) = 0, \qquad (16.19)$$

which says: The Lagrangian bracket symbols are in this case constant along the trajectory of each individual particle, a result that finds multiple applications in classical hydrodynamics (see, for example, H. LAMB [10]).

3. Derivation of a commutation relation from the Eulerian form of the hydrodynamical equations

Starting from the Eulerian form of the momentum equations for perfect fluids (16.3) and the continuity equation (16.4), the following commutation relation can be derived (H. ERTEL [1,2]; D. MORGENSTERN and I. SZABÓ [13], K. OSWATITSCH [14]):

$$\boxed{\begin{aligned}\left(\frac{d}{dt}\right)\left(\sigma\xi_k \frac{\partial}{\partial x_k}\right)\psi - \left(\sigma\xi_k \frac{\partial}{\partial x_k}\right)\left(\frac{d}{dt}\right)\psi \\ = \sigma\epsilon_{ijk}\frac{\partial p}{\partial x_i}\frac{\partial \sigma}{\partial x_j}\frac{\partial \psi}{\partial x_k}\end{aligned}}$$
$$(16.20)$$

where

$$\xi_k = \epsilon_{ijk}\frac{\partial v_j}{\partial x_i} \qquad (16.21)$$

are the components of the vorticity vector (curl of the velocity vector). This commutation relation (16.20), expressed in the Eulerian framework, has already been repeatedly used in special and generalized applications (H. ERTEL [3,4], H. FORTAK [6], G. HOLLMANN [7], P. MAUERSBERGER [12]).

4. The relation between the two commutation formulas

Since the derivation of (16.20) requires consideration of the continuity equation, an equation that was not necessary for the derivation of the commutation relation (16.14), the transformation of (16.14) to (16.20) and vice versa is of interest for discovering the connection between the two commutation relations (16.14) and (16.20).

In order to transform for example (16.14) into (16.20), one takes the product of (16.14) and (16.2), an operation which is equivalent to the introduction of the continuity equation (16.2) into (16.14). With consideration of the time-independence of σ_0, we obtain:

$$\left(\frac{\partial}{\partial t}\right)\left(\sigma_0 \{v, x\}_k \frac{\partial}{\partial a_k}\right)\psi$$
$$- \left(\sigma_0 \{v, x\}_k \frac{\partial}{\partial a_k}\right)\left(\frac{\partial}{\partial t}\right)\psi$$
$$= \sigma D^{-1}\epsilon_{ijk}\frac{\partial p}{\partial a_i}\frac{\partial \sigma}{\partial a_j}\frac{\partial \psi}{\partial a_k}, \quad (16.22)$$

the right side of which can be written, using (16.5) and (16.11), as

$$\sigma D^{-1}\epsilon_{ijk}\frac{\partial p}{\partial a_i}\frac{\partial \sigma}{\partial a_j}\frac{\partial \psi}{\partial a_k}$$
$$= \sigma\frac{\partial(p, \sigma, \psi)}{\partial(a_1, a_2, a_3)}\left(\frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)}\right)^{-1} \quad (16.23)$$

which in turn, with the use of the well-known functional determinant product rule, becomes

$$\sigma D^{-1}\epsilon_{ijk}\frac{\partial p}{\partial a_i}\frac{\partial \sigma}{\partial a_j}\frac{\partial \psi}{\partial a_k} = \sigma\frac{\partial(p, \sigma, \psi)}{\partial(x_1, x_2, x_3)}$$
$$= \sigma\epsilon_{ijk}\frac{\partial p}{\partial x_i}\frac{\partial \sigma}{\partial x_j}\frac{\partial \psi}{\partial x_k} \quad (16.24)$$

which is in agreement with the right side of (16.20).

On the left side of (16.20) the following rearrangement is possible:

$$\{v, x\}_k \frac{\partial}{\partial a_k} = \epsilon_{ijk}\frac{\partial v_m}{\partial a_i}\frac{\partial x_m}{\partial a_j}\frac{\partial}{\partial a_k}$$
$$= \epsilon_{ijk}\frac{\partial x_q}{\partial a_i}\frac{\partial v_m}{\partial x_q} \cdot \frac{\partial x_r}{\partial a_j}\frac{\partial x_m}{\partial x_r} \cdot \frac{\partial x_s}{\partial a_k}\frac{\partial}{\partial x_s}$$
$$= \epsilon_{ijk}\frac{\partial x_q}{\partial a_i}\frac{\partial x_r}{\partial a_j}\frac{\partial x_s}{\partial a_k} \cdot \frac{\partial v_m}{\partial x_q}\frac{\partial x_m}{\partial x_r}\frac{\partial}{\partial x_s}. \quad (16.25)$$

Using the general formula of the determinant symbolism (see, for example, H. JEFFREYS [8], H. JEFFREYS and B. SWIRLES [9], A. J. MCCONNELL [11], G. TEMPLE [15])

$$\epsilon_{ijk}\frac{\partial x_q}{\partial a_i}\frac{\partial x_r}{\partial a_j}\frac{\partial x_s}{\partial a_k} = \epsilon_{qrs}D, \qquad (16.26)$$

and the substitution tensor

$$\frac{\partial x_m}{\partial x_r} = \delta_{mr} \qquad (16.27)$$

in (16.25), we obtain, with consideration of (16.26), (16.21) and (16.2), the operator equation:

$$\begin{aligned} \sigma_0 \{v, x\}_k \frac{\partial}{\partial a_k} &= \sigma_0 D \epsilon_{qrs} \frac{\partial v_r}{\partial x_q} \frac{\partial}{\partial x_s} \\ &= \sigma \xi_s \frac{\partial}{\partial x_s} = \sigma \xi_k \frac{\partial}{\partial x_k}. \end{aligned} \qquad (16.28)$$

The substitution of (16.24) and (16.28) in (16.22) and the change from the Lagrangian to the Eulerian form of the material derivative $\partial/\partial t \to d/dt$ yields the commutation relation (16.20), q.e.d.

The inverse transformation, that is the elimination of the continuity equation from (16.20) to recover the commutation relation (16.14), has already discussed in another paper (H. Ertel [5]). Thus, using the continuity equation (16.2) with the functional determinant (16.5), a reversible connection between the two commutation relations (16.14) and (16.20) has been established. The transformation of the commutation relations (16.14) and (16.20) into one another is always executable, since the continuity equation, as an expression of the law of the conservation of fluid matter, provides a guarantee for it in that the functional determinant (16.5) is always different from zero.

References

[1] ERTEL, H., Über hydrodynamische Wirbelsätze. Physikal. Zeitschr., Leipzig, 43 Jahrg. (1942) 526–529.

[2] ERTEL, H., Ein neuer hydrodynamischer Wirbelsatz. Meteorol. Zeitschr., Braunschweig, 59. Jahrg. (1942) 277–281.

[3] ERTEL, H., Über das Verhältnis des neuen hydrodynamischen Wirbelsatzes zum Zirkulationsatz von V. BJERKNES. Meteorol. Zeitschr., Braunschweig, 59. Jahrg. (1942) 385–387.

[4] ERTEL, H., Ein neuer hydrodynamischer Erhaltungssatz. Die Naturwissenschaften, Berlin, 30. Jahrg. (1942) 543–544.

[5] ERTEL, H., Vertauschungs-Relationen der Hydrodynamik. Monatsber. Dt. Akad. Wiss., Berlin, Bd. 6 (1964) 838–841.

[6] FORTAK, H., Zur Frage allgemeiner hydrodynamischer Erhaltungssätze. Gerlands Beitr. z. Geophysik, Leipzig, Bd. 65 (1956) 283–294.

[7] HOLLMANN, G., Ein vollständiges System hydrodynamischer Erhaltungssätze. Archiv f. Meteorologie, Geophysik u. Bioklimatologie, Wien, Ser. A, Bd. 14 (1964) 1–13.

[8] JEFFREYS, H., Cartesian Tensors. Cambridge 1952 (Reprinted), 15.

[9] JEFFREYS, H., and B. SWIRLES, Methods of Mathematical Physics. Third Edition. Cambridge 1956, 72.

[10] LAMB, H., Hydrodynamics. First American Edition. New York 1945, 204.

[11] MCCONNELL, A. J., Applications of the Absolute Differential Calculus. London and Glasgow 1951 (Reprinted), 10–11.

[12] MAUERSBERGER, P., Zur Kinematik der magnetischen Feldlinien im Plasma. Pure and Applied Geophysics, Basel, Vol. 87 (1964/I) 143–148.

[13] MORGENSTERN, D., and I. SZABÓ, Vorlesungen über Theoretische Mechanik. Berlin-Göttingen-Heidelberg 1961, 182.

[14] OSWATITSCH, K., Physikalische Grundlagen der Strömungslehre. In: Handbuch der Physik, herausg. von S. FLÜGGE. Bd. VIII/1. Berlin-Göttingen-Heidelberg 1959, 43–45.

[15] TEMPLE, G., Cartesian Tensors. London–New York 1960, 18.

[16] WHITTAKER, E. F., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Fourth Edition. Cambridge 1952, 298.

17 ERTEL (1965b): Commutative operators for unsteady flow of a perfect, piezotropic fluid

The flow of a perfect, piezotropic fluid (in the notation of, for example, J. SERRIN [7]) is governed by the momentum equation

$$\frac{dv_i}{dt} = -\frac{\partial}{\partial x_i}\left(\Phi + \int \sigma\, dp\right), \qquad (17.1)$$

or equivalently,

$$\frac{\partial v_i}{\partial t} = \epsilon_{ijk}v_j\xi_k - \frac{\partial H}{\partial x_i}, \qquad (17.2)$$

by the continuity equation

$$\frac{d\sigma}{dt} = \sigma\frac{\partial v_j}{\partial x_j}, \qquad (17.3)$$

and by the piezotropic relation

$$\sigma = \sigma(p), \qquad (17.4)$$

where x_i $(i = 1, 2, 3)$ = orthogonal cartesian coordinates (right-handed system), v_i $(i = 1, 2, 3)$ = components of the velocity vector $\mathbf{v}$, ϵ_{ijk} $(i, j, k = 1, 2, 3)$ = permutation symbol (antisymmetric in all indices), and $\xi_k = \epsilon_{ijk}\frac{\partial v_j}{\partial x_i}$ = components of the vorticity vector $\boldsymbol{\xi} = \nabla \times \mathbf{v}$. The scalars $\Phi, \sigma, p, H = \frac{1}{2}v^2 + \Phi + \int \sigma\, dp$ (with $v^2 = v_j v_j$) and t respectively denote the potential for external forces, the specific volume (reciprocal of density), the pressure, the energy, and the time; in addition

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j} \qquad (17.5)$$

is the material differential operator of hydrodynamics. In the previous equations we consistly use the usual formulation of vector analysis (see, for example, G. H. A. COLE [1], J. O. HINZE [5]) with the Einstein summation convention. From equations (17.1), or equivalently (17.2), and (17.3), we deduce that the operators

$$\frac{d}{dt} \qquad \text{and} \qquad \sigma\xi_i\frac{\partial}{\partial x_i} \qquad (= \sigma\boldsymbol{\xi} \cdot \nabla)$$

applied to the twice differentiable function $\psi = \psi(x_1, x_2, x_3, t)$ are commutative (H. ERTEL [2], [3]; see, for example, D. MORGENSTERN and I. SZABÓ [6]), so that

$$\left(\frac{d}{dt}\right)\left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)\psi - \left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)\left(\frac{d}{dt}\right)\psi = 0. \qquad (17.6)$$

We wish to utilize the commutation relation (17.6) to prove that the operators

$$\frac{d}{dt} \qquad \text{and} \qquad \sigma\xi_i\frac{\partial}{\partial t} \qquad \left(= \sigma\boldsymbol{\xi}\frac{\partial}{\partial t}\right)$$

are commutative when applied to the components of the flow velocity vector

$$\mathbf{v} = \{v_i\} \qquad (i = 1, 2, 3)$$

for the flow field of a perfect, piezotropic fluid:

$$\boxed{\left(\frac{d}{dt}\right)\left(\sigma\xi_i\frac{\partial}{\partial t}\right)v_i - \left(\sigma\xi_i\frac{\partial}{\partial t}\right)\left(\frac{d}{dt}\right)v_i = 0,}$$
$$(17.7)$$

or in the coordinate-independent formulation of vector analysis:

$$\boxed{\left(\frac{d}{dt}\right)\left(\sigma\boldsymbol{\xi}\frac{\partial}{\partial t}\right)\mathbf{v} - \left(\sigma\boldsymbol{\xi}\frac{\partial}{\partial t}\right)\left(\frac{d}{dt}\right)\mathbf{v} = 0.}$$
$$(17.8)$$

To prove (17.7), or equivalently (17.8), we set $\psi = H$ in (17.6), which results in

$$\left(\frac{d}{dt}\right)\left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)H - \left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)\left(\frac{d}{dt}\right)H = 0. \qquad (17.9)$$

From the momentum equation in the form (17.2) it follows that

$$\left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)H = -\left(\sigma\xi_i\frac{\partial}{\partial t}\right)v_i, \qquad (17.10)$$

whereas from the momentum equation in the form (17.1) it follows that

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = -\left(v_i\frac{\partial}{\partial x_i}\right)\left(\Phi + \int \sigma\, dp\right) \qquad (17.11)$$

and, with consideration of (17.5) and the definition of the total energy H, that:

$$\frac{dH}{dt} = \frac{\partial}{\partial t}\left(\Phi + \int \sigma\, dp\right). \qquad (17.12)$$

From this it immediately follows that

$$\left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)\left(\frac{d}{dt}\right)H$$
$$= \left(\sigma\xi_i\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x_i}\left(\Phi + \int \sigma\, dp\right), \qquad (17.13)$$

or with the use of (17.1):

$$\left(\sigma\xi_i\frac{\partial}{\partial x_i}\right)\left(\frac{d}{dt}\right)H = -\left(\sigma\xi_i\frac{\partial}{\partial t}\right)\left(\frac{d}{dt}\right)v_i. \tag{17.14}$$

Substitution of (17.10) and (17.14) into (17.9) yields the desired equation (17.7), thereby completing our argument and demonstrating the coordinate-free notation of equation (17.8). It is possible to obtain equations (17.7) and (17.8) by another method, based on the hydrodynamical vorticity equation, in which yet another operation can be given, with the application to the flow field of a perfect, piezotropic fluid yielding the commutation relation if the field is stationary (H. ERTEL [4]).

References

[1] COLE, G. H. A., Fluid Dynamics. London-New York, 1962, 3.

[2] H. ERTEL, Über hydrodynamische Wirbelsätze. Phys. Z., Leipzig, 43. Jahrg., 526 (1942).

[3] H. ERTEL, Hydrodynamische Vertauschungs-Relationen. Acta Hydrophysica, Berlin, Bd. IX, 115 (1965).

[4] H. ERTEL, Kommutative Operatoren der Geschwindigkeitsfelder perfekter, piezotroper Flüssigkeiten. Acta Hydrophysica, Berlin, Bd. X, Heft 2 (1965); z. Z. im Druck.

[5] HINZE, J. O., Turbulence. New York-Toronto-London 1959, 15.

[6] MORGENSTERN, D., and I. SZABÓ, Vorlesungen über Theoretische Mechanik. Berlin-Göttingen-Heidelberg 1961, 182.

[7] SERRIN, J., Mathematical Principles of Classical Fluid Mechanics. In: Handbuch der Physik, herausgegeben von S. FLÜGGE. Band VIII/1. Berlin-Göttingen-Heidelberg 1959, 150.

18   ERTEL (1965c): Theorem on unimodular transformations of the hydrodynamical labeling coordinates

Summary

From the equations which express the dependence of the initial coordinates upon the coordinates of the space-time continuum we can form an invariant with respect to unimodular transformations from one system of initial coordinates to another.

1. Introduction

Assume that a fluid flows through an n-dimensional space. A particular point in space is given by the coordinates $\{x_1, x_2, \ldots, x_n\}$, measured with respect to a fixed frame of reference, and a particular particle is identified by the parameters $\{a_1, a_2, \ldots, a_n\}$ ("labeling coordinates," see, for example, V. BJERKNES, J. BJERKNES, H. SOLBERG, T. BERGERON [1], H. L. DRYDEN, F. D. MURNAGHAN, H. BATEMAN [2]). An example of labeling coordinates are the particle positions at $t = 0$ ("Anfangs-coordinates"). Special presuppositions on the nature of the fluid (whether ideal or viscous, incompressible or compressible) are not necessary for our purposes.

For brevity the notation of tensor analysis is adopted. All indices range from 1 to n; a term with repeated indices is to be summed from 1 to n. Other symbols are defined in the text.

2. Statement of the theorem

The solutions of the hydrodynamical equations, including the required physical connections, are embodied in the system of implicit functions ($j = 1, 2, \ldots, n$):

$$R_j(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_n, t) = 0. \tag{18.1}$$

Solving for x_j gives the parameter description of particle trajectories:

$$x_j = x_j(a_1, a_2, \ldots, a_n, t), \tag{18.2}$$

whereas solution for a_j results in:

$$a_j = a_j(x_1, x_2, \ldots, x_n, t), \tag{18.3}$$

from which, for example for the purposes of meteorology, one can answer the question of which particle (air mass) passes through the fixed spatial coordinates $\{x_1, x_2, \ldots, x_n\}$ at time t.

In that the functions (18.2) are to be regarded as continuous and differentiable, and furthermore that the continuity equation in Lagrangian form requires that the functional determinant of $x_1, x_2, \ldots, x_n, t$ with respect to $a_1, a_2, \ldots, a_n, t$, or its inverse, is always different than zero, the system (18.2) is the inverse transformation of (18.2) (and vice versa).

A system of infinitesimal variations $\delta x_1, \delta x_2, \ldots,$ $\delta x_n, \delta t$ of the coordinates of space-time $x_1, x_2, \ldots,$ x_n, t is related to the variations of the labeling coordinates by

$$\delta a_j = \frac{\partial a_j}{\partial x_k} \delta x_k + \frac{\partial a_j}{\partial t} \delta t \qquad (j, k = 1, 2, \ldots, n). \tag{18.4}$$

Here the δx_k $(k = 1, 2, \ldots, n)$ and δt are independently and freely selectable. The operator $\partial/\partial t$ denotes the local time derivative in the Eulerian sense, i.e.,

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t} \right)_{x_1, x_2, \ldots, x_n} \tag{18.5}$$

not the operator for individual or material differentiation with respect to time:

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} \right)_{a_1, a_2, \ldots, a_n} \tag{18.6}$$

which makes possible simple expressions for preservation of the particle-bound properties (H. ERTEL [3], G. HOLLMANN [4]). For example, the individual preservation of the labeling coordinates is expressed by the system

$$\frac{da_j}{dt} = 0 \qquad (j = 1, 2, \ldots, n) \tag{18.7}$$

while $\partial a_j / \partial t$ for a moving fluid can never disappear because of the local particle change at fixed spatial points.

According to (18.3), the partial derivatives $\partial a_j / \partial x_k$ and $\partial a_j / \partial t$ in (18.4) are functions of $x_1, x_2, \ldots, x_n$ and t. However, by means of (18.2), they can also be represented as functions of $a_1, a_2, \ldots, a_n$ and t. This latter representation is assumed for the purpose of the following calculations.

In place of the system a_j $(j = 1, 2, \ldots, n)$ we now transform to another system b_j $(j = 1, 2, \ldots, n)$ of labeling coordinates through the transformation

$$b_j = b_j(a_1, a_2, \ldots, a_n) \qquad (j = 1, 2, \ldots, n) \tag{18.8}$$

with the transformation determinant

$$\frac{\partial(b_1, b_2, \ldots, b_n)}{\partial(a_1, a_2, \ldots, a_n)} = 1 \tag{18.9}$$

(unimodular transformation; see, for example, B. J. NAAS and H. L. SCHMID [6], H. REICHARDT [7], P. A. SCHIROKOW and A. P. SCHIROKOW [8], K. STRUBECKER [9], E. VIDAL ABASCAL [10]). Then from (18.3) and (18.4), with the aid of (18.8) and (18.9), we deduce the following theorem:

The form

$$F = \frac{\partial \, \delta a_j}{\partial a_j} \tag{18.10}$$

is an invariant with respect to a unimodular transformation from one system of labeling coordinates to another.

3. Proof of the theorem

The variations (18.4) are related through (18.8) to the variations

$$\delta b_j = b_{j,k} \, \delta a_k \tag{18.11}$$

of the new labeling coordinates, where $\partial b_j / \partial a_k = b_{j,k}$.

Identifying b_{jm} (without the comma between the indices) as the subdeterminant corresponding to the element $b_{j,m}$ of the determinant (18.9), the determinant adopts the form:

$$b_{jm} b_{k,m} = b_{mj} b_{m,k} = \begin{cases} +1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} \tag{18.12}$$

Then it can be shown that the solutions of the system (18.11) for δa_j can be written as

$$\delta a_j = b_{kj} \, \delta b_k, \tag{18.13}$$

from which the expression in the above theorem can be written as

$$\frac{\partial \, \delta a_j}{\partial a_j} = b_{kj,j} \, \delta b_k + b_{kj} \frac{\partial \, \delta b_k}{\partial a_j}. \tag{18.14}$$

Through alteration of the summation indices on the right hand side, we obtain

$$\frac{\partial \, \delta a_j}{\partial a_j} = b_{jk,k} \, \delta b_j + b_{jm} \frac{\partial \, \delta b_j}{\partial a_m}. \tag{18.15}$$

In (18.15) we have

$$b_{jk,k} = \frac{\partial b_{jk}}{\partial a_k} = \frac{\partial b_{jk}}{\partial b_{rs}} b_{r,sk} = 0, \tag{18.16}$$

because $b_{r,sk} = \partial^2 b_r / \partial a_k \partial a_s = \partial^2 b_r / \partial a_s \partial a_k$ is symmetric in the indices s and k, and the $(n-2)$-order subdeterminant of the determinant (18.9)

$$\frac{\partial b_{jk}}{\partial b_{r,s}} = -\frac{\partial b_{js}}{\partial b_{r,k}} \tag{18.17}$$

is antisymmetric in these indices (see, for example, B. E. MADELUNG [5]). Thus, (18.15) reduces to

$$\frac{\partial \, \delta a_j}{\partial a_j} = b_{jm} \frac{\partial \, \delta b_j}{\partial a_m}. \tag{18.18}$$

47

Analogous to equations (18.3) and (18.4) for the labeling coordinates a_j and their variations δa_j with respect to the coordinates $(x_1, x_2, \ldots, x_n, t)$ of the space-time continuum, for the label coordinates b_j and their variations δb_j we have

$$b_j = b_j(x_1, x_2, \ldots, x_n, t), \qquad (18.19)$$

$$\delta b_j = \frac{\partial b_j}{\partial x_k} \delta x_k + \frac{\partial b_j}{\partial t} \delta t. \qquad (18.20)$$

By elimination of the spatial coordinates $x_1, x_2, \ldots, x_n$ the coefficients $\partial b_j / \partial x_k$ and $\partial b_j / \partial t$ can be expressed as functions of $b_1, b_2, \ldots, b_n$ and t. Then, the right hand side of (18.18) can be transformed into

$$b_{jm} \frac{\partial \, \delta b_j}{\partial a_m} = b_{jm} b_{k,m} \frac{\partial \, \delta b_j}{\partial b_k} \qquad (18.21)$$

so that from (18.18), with consideration of (18.12), we obtain

$$\boxed{\frac{\partial \, \delta a_j}{\partial a_j} = \frac{\partial \, \delta b_j}{\partial b_j}} \qquad (18.22)$$

which proves the invariance of the form (18.10) with respect to the unimodular transformation (18.8) and (18.9) of the labeling coordinates.

4. Corollary

From (18.22) there follows the invariance relations

$$\boxed{\frac{\partial}{\partial a_j}\left(\frac{\partial a_j}{\partial x_i}\right) = \frac{\partial}{\partial b_j}\left(\frac{\partial b_j}{\partial x_i}\right)} \qquad (i = 1, 2, \ldots, n)$$

$$(18.23)$$

and

$$\boxed{\frac{\partial}{\partial a_j}\left(\frac{\partial a_j}{\partial t}\right) = \frac{\partial}{\partial b_j}\left(\frac{\partial b_j}{\partial t}\right)} \qquad (18.24)$$

with respect to unimodular transformation of the labeling coordinates, because the variations $\delta x_1, \delta x_2, \ldots, \delta x_n, \delta t$ are arbitrary and independent.

5. Hydrodynamical interpretation

The derived relations (18.10), or equivalently (18.22), (18.23) and (18.24) (especially, equation (18.24)) require additional explanation.

In (18.24) $\partial a_j / \partial t$ is the local time derivative of the labeling coordinate a_j at the fixed point $\{x_1, x_2, \ldots, x_n\}$ of reference space, which arises due to the passage of labeled particles. If these changes are assigned to the particle labels that occur at the same time t at neighboring spatial points, and if with these particle

labels the expression on the left side of (18.24) is calculated, then the same result is invariably obtained in another system of labeling coordinates b_j that is related to the system a_j by a unimodular tranformation.

The interpretation of the formula (18.23) is similar to the interpretation of (18.10) and (18.22). The creation of these invariant forms is based on the hydrodynamical fact that all systems of labeling coordinates must contain the necessary number of eliminable coordinates (here $x_1, x_2, \ldots, x_n$).

The invariant form F in equation (18.10) is in general a function of the coordinates of the space-time continuum, which depends on the physical behaviour of the fluid. If the labeling coordinates a_j are selected to be the particle positions at time $t = 0$, and the fluid is incompressible, then $F = 0$, as can be seen from the continuity equation in Lagrangian form. Then, in equations (18.22), (18.23) and (18.24) the sums on either side vanish.

In the case of a compressible fluid, we obtain

$$\boxed{F = \frac{\partial \, \delta a_j}{\partial a_j} = \delta\left\{\ln\left(\frac{\rho}{\rho_0}\right)\right\}} \qquad (18.25)$$

(ρ =density, ρ_0 =initial density). This can be proved through logarithmic variation of the continuity equation

$$\frac{\partial(a_1, a_2, \cdots, a_n)}{\partial(x_1, x_2, \cdots, x_n)} = \frac{\rho}{\rho_0} \qquad (18.26)$$

under the condition that the operators δ and $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ are commutable.

References

[1] BJERKNES, V., BJERKNES, J., SOLBERG, H., u. T. BERGERON, Physikalische Hydrodynamik mit Anwendung auf die Dynamische Meteorologie, Berlin 1933, 51–53.

[2] DRYDEN, H. L., MURNAGHAN, F. D., and H. BATEMAN, Hydrodynamics, New York 1956, 32.

[3] ERTEL, H., Teorema sobre invariantes sustanciales de la Hidrodinámica. Gerl. Beitr. Geophys. *69* (1960), 290–293.

[4] HOLLMANN, G., Ein vollständiges System hydrodynamischer Erhaltungssätze. Arch. Meteorol. Geophys. Bioklim. A *14* (1964), 1–13.

[5] MADELUNG, E., Eie mathematischen Hilfsmittel des Physikers, 7. Aufl. Berlin/Göttingen/Heidelberg 1964, 145.

[6] NAAS, J., u. H. L. SCHMID, Mathematisches Wörterbuch, Bd. II, Berlin/Leipzig 1961, 787.

[7] REICHARDT, H., Vorlesungen über Vektor- und Tensorrechnung, Berlin 1957, 29.

[8] SCHIROKOW, P. A., u. A. P. SCHIROKOW, Affine Differentialgeometrie, Leipzig 1962, 142.

[9] STRUBECKER, K., Differentialgeometrie II, Berlin 1958, 117–118.

[10] VIDAL ABASCAL, E., Introducción a la Geometría Diferencial, Madrid 1956, 277.

19 ERTEL (1970a): A relationship between kinematical parameters of horizontal flow fields in the atmosphere

Summary

The quasilinear inertia terms in two different forms are compatible only if the determinant of the coefficients of the velocity components is equal to zero. This condition of compatibility gives a relationship between the parameters which characterize the kinematics of two-dimensional air movements. The simple example of the gradient wind discloses the meaning of this relationship.

"Observations show that every large-scale current in the atmosphere is nearly horizontal" (*J. Holmboe, G. E. Forsythe, W. Gustin* [5]). The research methods of the kinematics of horizontal flow must develop integrating components of dynamic and synoptic meteorology (*V. Bjerknes, Th. Hesselberg, O. Devik* [1]; *V. Bjerknes, J. Bjerknes, H. Solberg, T. Bergeron* [2]; *J. M. Jansa Guardiola* [6]). That part of a horizontal flow field that is in the immediate vicinity of any moving air (mass) parcel can be regarded with sufficient approximation as evenly varying. For such regions, a relation between various parameters of the fields of flow will be derived here. For kinematic parameters this relation contains, on the one hand, the three components of the symmetrical deformation tensor and the spin of the air mass elements, and on the other hand, the streamline-tangential derivatives of the isotachs and isogons.

Let x and y be arbitrarily oriented orthogonal cartesian coordinates in the horizontal plane, which, together with an upward vertical z–coordinate, form a right-handed system. Based on this system, v_x and v_y are the horizontal components of the flow velocity. We define dv_x/dt and dv_y/dt as the material derivatives, with $\partial v_x/\partial t$ and $\partial v_y/\partial t$ the local derivatives of the flow components with respect to time, t, such that

$$\frac{dv_x}{dt} - \frac{\partial v_x}{\partial t} = \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y \qquad (19.1)$$

and

$$\frac{dv_y}{dt} - \frac{\partial v_y}{\partial t} = \frac{\partial v_y}{\partial x}v_x + \frac{\partial v_y}{\partial y}v_y \qquad (19.2)$$

are the nonlinear terms, those which generally contain the nonsymmetric gradient tensor (*D. Morgenstern and I. Szabó* [8])

$$V = \begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{pmatrix}. \qquad (19.3)$$

In addition, we use polar coordinates to represent the flow field, with positive definite $v = (v_x^{\,2} + v_y^2)^{1/2}$ representing the flow speed and α being the positive azimuthal angle, so that

$$v_x = v\cos\alpha, \qquad v_y = v\sin\alpha. \qquad (19.4)$$

The lines $v = $ const and $\alpha = $ const are the isotachs and isogons, respectively, of the flow field and $\partial/\partial s = \cos\alpha\,(\partial/\partial x) + \sin\alpha\,(\partial/\partial y)$ is the streamline-tangential derivative, which with (19.1) and (19.2) implies

$$\frac{dv_x}{dt} - \frac{\partial v_x}{\partial t} = v\frac{\partial v_x}{\partial s}, \qquad \frac{dv_y}{dt} - \frac{\partial v_y}{\partial t} = v\frac{\partial v_y}{\partial s}. \quad (19.5)$$

Because, in consideration of (19.4),

$$\frac{\partial v_x}{\partial s} = \frac{\partial v}{\partial s}\cos\alpha - v\frac{\partial\alpha}{\partial s}\sin\alpha, \qquad (19.6)$$

and

$$\frac{\partial v_y}{\partial s} = \frac{\partial v}{\partial s}\sin\alpha + v\frac{\partial\alpha}{\partial s}\cos\alpha, \qquad (19.7)$$

we can write expressions for the convection of v_x and v_y:

$$v\frac{\partial v_x}{\partial s} = \frac{\partial v}{\partial s}v_x - v\frac{\partial\alpha}{\partial s}v_y, \qquad (19.8)$$

$$v\frac{\partial v_y}{\partial s} = v\frac{\partial\alpha}{\partial s}v_x + \frac{\partial v}{\partial s}v_y, \qquad (19.9)$$

which with (19.5) results in

$$\frac{dv_x}{dt} - \frac{\partial v_x}{\partial t} = \frac{\partial v}{\partial s} v_x - v \frac{\partial \alpha}{\partial s} v_y, \qquad (19.10)$$

$$\frac{dv_y}{dt} - \frac{\partial v_y}{\partial t} = v \frac{\partial \alpha}{\partial s} v_x + \frac{\partial v}{\partial s} v_y. \qquad (19.11)$$

Subtraction of equation (19.10) from (19.1) and of equation (19.11) from (19.2) yields the homogeneous system, expressed in terms of both the flow components (v_x, v_y):

$$\left(\frac{\partial v_x}{\partial x} - \frac{\partial v}{\partial s} \right) v_x + \left(\frac{\partial v_x}{\partial y} + v \frac{\partial \alpha}{\partial s} \right) v_y = 0, \quad (19.12)$$

$$\left(\frac{\partial v_y}{\partial x} - v \frac{\partial \alpha}{\partial s} \right) v_x + \left(\frac{\partial v_y}{\partial y} - \frac{\partial v}{\partial s} \right) v_y = 0, \quad (19.13)$$

which, as evident from its derivation, must apply for nonstationary as well as stationary flow fields.

We now go from the nonsymmetric gradient tensor to the symmetric deformation tensor (tensor of the deformation of the flow)

$$D = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \qquad (19.14)$$

by defining the following components:

$$\begin{aligned} D_{xx} &= \frac{\partial v_x}{\partial x}, \\ D_{yy} &= \frac{\partial v_y}{\partial y}, \\ D_{xy} &= D_{yx} = \frac{1}{2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right), \end{aligned} \qquad (19.15)$$

and using the identities:

$$\frac{\partial v_x}{\partial y} = D_{xy} - \frac{1}{2} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) = D_{xy} - S, \quad (19.16)$$

$$\frac{\partial v_y}{\partial x} = D_{xy} + \frac{1}{2} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) = D_{xy} + S, \quad (19.17)$$

where the spin (*A. S. Ramsey* [9]; *O. G. Sutton* [11])

$$S = \frac{1}{2} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \qquad (19.18)$$

denotes the turning of an air (mass) parcel around the vertical z-axis.

The components D_{xx} and D_{yy} of the deformation tensor (19.14) describe the stretching and compression rates of the fluid parcels, while the component D_{xy} expresses a shearing rate (*O. Tietjens* [12]). A necessary and sufficient condition for motion without deformation, analogous to a rigid body, is that all terms in the deformation tensor vanish (*J. Serrin* [10]).

The spin defined in (19.18) is the turning rate of air parcels relative to the (x, y)-system. Relative to the convection of the direction of flow α the spin of a fluid particle is given by

$$R = S - v \frac{\partial \alpha}{\partial s}. \qquad (19.19)$$

Now, through elimination of the components of the gradient tensor of the flow in equations (19.12) and (19.13), and by means of (19.15), (19.16), and (19.17) the following system results:

$$\left(D_{xx} - \frac{\partial v}{\partial s} \right) v_x + \left(D_{xy} - S + v \frac{\partial \alpha}{\partial s} \right) v_y = 0, \qquad (19.20)$$

$$\left(D_{xy} + S - v \frac{\partial \alpha}{\partial s} \right) v_x + \left(D_{yy} - \frac{\partial v}{\partial s} \right) v_y = 0, \qquad (19.21)$$

or, considering (19.19), in a somewhat more abbreviated form:

$$\left(D_{xx} - \frac{\partial v}{\partial s} \right) v_x + (D_{xy} - R) v_y = 0, \qquad (19.22)$$

$$(D_{xy} + R) v_x + \left(D_{yy} - \frac{\partial v}{\partial s} \right) v_y = 0. \qquad (19.23)$$

The necessary and sufficient condition (*M. Bocher* [3]) for the compatability of equations (19.22) and (19.23) appears in determinant form as:

$$\begin{vmatrix} D_{xx} - \dfrac{\partial v}{\partial s} & D_{xy} - R \\ D_{xy} + R & D_{yy} - \dfrac{\partial v}{\partial s} \end{vmatrix} = 0, \qquad (19.24)$$

that is, the equation

$$\boxed{ \begin{aligned} \left(\frac{\partial v}{\partial s} \right)^2 &- \frac{\partial v}{\partial s} \left(D_{xx} + D_{yy} \right) \\ &+ \begin{vmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{vmatrix} + R^2 = 0 \end{aligned} } \qquad (19.25)$$

must be satisfied.

Equation (19.25) represents the sought-after relation between the kinematic parameters—their invariance in relation to rotations of the orthogonal (x, y) coordinate system about the vertical $z-$axis is thereby assured, insomuch as it is well established that the trace $D_{xx} + D_{yy}$ and the determinant

$$|D| = \begin{vmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{vmatrix} \qquad (19.26)$$

of the deformation tensor (19.14) are invariant (*G. Joos and Th. Kaluza* [7]; *W. Weizel* [13],[14]).

By means of the relation (19.25), it is simple to examine which horizontal flows in the atmosphere lead to deformationless movement of individual air parcels. Since in this case $D_{xx} = D_{yy} = D_{xy} = 0$ must apply, (19.25) reduces to the equation

$$\left(\frac{\partial v}{\partial s}\right)^2 + R^2 = 0, \qquad (19.27)$$

which involves the two conditions:

$$\frac{\partial v}{\partial s} = 0 \qquad (19.28)$$

and

$$R = S - v\frac{\partial \alpha}{\partial s} = 0. \qquad (19.29)$$

The first condition (19.28) is fulfilled by a stationary flow gradient as a consequence of flow parallel to the isobars (*A. Defant and F. Defant* [4]). Assuming that the air parcel motions are deformationless, the second condition (19.29) means that the spin S must agree with the convection of the flow direction; in the special case of geostrophic flow ($\partial \alpha/\partial s = 0$), $S = 0$.

Consider when the geostrophic flow blows in the x-direction, with shear due to the change of the wind velocity v_x in the gradient-parallel y-direction, thus the air parcels possess a spin S, its magnitude agreeing with the shear, whereby the signs of S and D_{xy} are opposite. In consideration of (19.29), the condition (19.25) now demands

$$-D_{xy}^2 + S^2 = (S + D_{xy})(S - D_{xy}) = 0, \quad (19.30)$$

which is satisfied since $D_{xy} = \frac{1}{2}(\partial v_x/\partial y)$ from (19.15) and $S = -\frac{1}{2}(\partial v_x/\partial y)$ from (19.18) and $S + D_{xy} = 0$ identically. The spin is negative (anticyclonic) with positive shearing, and vice versa.

Consider the deformation of a horizontal square into a rhombus by the shearing effect $\frac{1}{2}(\partial v_x/\partial y)$. This operation turns the coordinate system of the original orthogonal diagonals in such a way that the orthogonality of the diagonals is conserved, according to a theorem of geometry. This turning is a simple example of the spin.

References

[1] *Bjerknes, V., Th. Hesselberg, O. Devik:* Dynamic Meteorology and Hydrography. Part II, Kinematics. Washington 1911.

[2] *Bjerknes, V., J. Bjerknes, H. Solberg, T. Bergeron:* Physikalische Hydrodynamik mit Anwendungen auf die dynamische Meteorologie. Berlin 1933.

[3] *Bocher, M.:* Introduction to Higher Algebra. New York 1964, p. 47.

[4] *Defant, A., und F. Defant:* Physikalische Dynamik der Atmosphäre. Frankfurt am Main, 1958, S. 121.

[5] *Holmboe, J., G. E. Forsythe and W. Gustin:* Dynamic Meteorology. Second Printing. New York (London) 1948, p. 173.

[6] *Jansa Guardiola, J. M.:* Tratado de Meteorología Teórica. Tomo II. Estática–Cinemática. Madrid 1960, pag. 152.

[7] *Joos, G., und Th. Kaluza:* Höhere Mathematik für den Praktiker. Sechste verbesserte Auflage. Leipzig 1952, S. 118.

[8] *Morgenstern, D., und I. Szabó:* Vorlesungen über Theoretische Mechanik. Berlin (Göttingen) Heidelberg 1961, S. 79.

[9] *Ramsey, A. S.:* A Treatise on Hydromechanics. Part II. Hydrodynamics. Fourth Edition. Reprinted. London 1947, p. 27.

[10] *Serrin, J.:* Mathematical Principles of Classical Fluid Mechanics. Handbuch der Physik, Bd. III/1. Berlin (Göttingen) Heidelberg 1959, p. 139.

[11] *Sutton, O. G.:* Micrometeorology. A Study of Physical Processes in the Lowest Layers of the Earth's Atmosphere. London 1953, p. 23.

[12] *Tietjens, O.:* Hydro- und Aeromechanik nach Vorlesungen von *L. Prandtl*. Erster Band. Zweite Auflage. Wien 1944, S. 75.

[13] *Weizel, W.:* Physikalische Formelsammlung. Erster Band. Mannheim 1962, S. 48.

[14] *Weizel, W.:* Lehrbuch der Theoretischen Physik. Erster Band. Physik der Vorgänge. Berlin (Göttingen) Heidelberg 1963, S. 142.

20 ERTEL (1970b): Spin and deformation tensor in connection with isogons and isotachs in two-dimensional streaming fields

Summary

In two-dimensional streams the acceleration can be expressed by isogons and by equivelocity contours, but it can be demonstrated also as a function of the two-dimensional deformation velocity and of the tensor of spin of fluid particles. On the basis of the condition of compatibility a connection of general validity can be obtained between all four mentioned quantities for both cases. That connection is quite independent both from the kind of fluid and from the characteristics in time of the streaming field.

1. Introduction

In recent years, the field of hydrodynamics has adopted a remarkable development by introducing parameters from differential geometry (see, for example, H. FORTAK [2], M. OLBERG [3]). Continuing in this vein, the present work links elements of the kinematics on large scales (macro-kinematic parameters: Isotachs and Isogons) with elements of the kinematics on small scales (micro-kinematic parameters: Spin and components of the deformation rate of the individual fluid particles).

2. Symbols

From the compatibility condition (H. ERTEL [1])

$$\begin{vmatrix} D_{xx} - \dfrac{\partial v}{\partial s} & D_{xy} - S + v\dfrac{\partial \alpha}{\partial s} \\[2ex] D_{xy} + S - v\dfrac{\partial \alpha}{\partial s} & D_{yy} - \dfrac{\partial v}{\partial s} \end{vmatrix} = 0, \quad (20.1)$$

we get two forms of the convective acceleration of two-dimensional flow fields, where, with an orthogonal cartesian coordinate system (x, y), the following symbols are used:

$$\begin{aligned} D_{xx} &= \frac{\partial v_x}{\partial x}, \\ D_{yy} &= \frac{\partial v_y}{\partial y}, \\ D_{xy} = D_{yx} &= \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right), \end{aligned} \quad (20.2)$$

which are the components of the deformation rate tensor associated with the velocity components (v_x, v_y). Furthermore, a fluid particle's spin (for example, O. G. SUTTON [5]) is

$$S = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right), \quad (20.3)$$

the magnitude of the flow velocity is

$$v = \left(v_x^2 + v_y^2\right)^{\frac{1}{2}}, \quad (20.4)$$

and the isogon angle, considered positive with counterclockwise rotation with respect to the x-axis, is

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right). \quad (20.5)$$

The curves $v =$const are the isotachs and the curves $\alpha =$const are the isogons (isoclines). The operator $\partial/\partial s$ in (20.1) is the derivative along the streamline. The derivative in the perpendicular direction, in the sense of counterclockwise rotation, is designated as $\partial/\partial n$, so one can write

$$D_{xx} + D_{yy} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial v}{\partial s} + v\frac{\partial \alpha}{\partial n} \quad (20.6)$$

and

$$2S = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{\partial v}{\partial n} + v\frac{\partial \alpha}{\partial s}, \quad (20.7)$$

with the aid of

$$v_x = v\cos\alpha, \qquad v_y = v\sin\alpha \quad (20.8)$$

and

$$\begin{aligned} \frac{\partial}{\partial s} &= \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial n} &= -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}. \end{aligned} \quad (20.9)$$

3. A relation between macro- and micro-kinematical parameters in the flow field

Instead of introducing three separate components of the deformation rate tensor, it is convenient to form a microkinematic parameter which is a combination of the components, namely the determinant

$$D = \begin{vmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{vmatrix}. \quad (20.10)$$

Such a quantity has the property of invariance under coordinate transformations (e.g., W. WEIZEL [6]). Then, it is a matter of developing a relationship between the two macrokinematic parameters (v, α) and the two microkinematic parameters (S, D).

The derivation of this relationship is then very simple. Expanding the compatibility determinant (20.1), the equation

$$\left(\frac{\partial v}{\partial s}\right)^2 - \frac{\partial v}{\partial s}\left(D_{xx} + D_{yy}\right)$$
$$+ D + \left(S - v\frac{\partial \alpha}{\partial s}\right)^2 = 0 \quad (20.11)$$

is obtained. First, (20.6) is substituted into (20.11):

$$-v\frac{\partial \alpha}{\partial n}\frac{\partial v}{\partial s} + D + \left(S - v\frac{\partial \alpha}{\partial s}\right)^2 = 0. \quad (20.12)$$

Then, substituting the expression given for $2S$ in (20.7) into

$$\left(S - v\frac{\partial \alpha}{\partial s}\right)^2 = S^2 - 2Sv\frac{\partial \alpha}{\partial s} + v^2\left(\frac{\partial \alpha}{\partial s}\right)^2 \quad (20.13)$$

yields

$$\left(S - v\frac{\partial \alpha}{\partial s}\right)^2 = S^2 + v\frac{\partial \alpha}{\partial s}\frac{\partial v}{\partial n}, \quad (20.14)$$

which then transforms (20.12) into

$$v\left(\frac{\partial \alpha}{\partial s}\frac{\partial v}{\partial n} - \frac{\partial \alpha}{\partial n}\frac{\partial v}{\partial s}\right) + D + S^2 = 0, \quad (20.15)$$

the sought-after equation that represents the relationship between v, α, D, and S. Using the orthogonal cartesian coordinates (x, y) instead of the natural coordinates (s, n), one can also write

$$v\left(\frac{\partial \alpha}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial v}{\partial x}\right) + D + S^2 = 0, \quad (20.16)$$

since the terms in parentheses in (20.15) and (20.16) are invariant under rotations of the two-dimensional coordinate system about the normal z-direction ($x, y, z =$ coordinate system). Applying the symbols of vector analysis, the equations (20.15, 20.16) take on the form

$$v\left(\nabla \alpha \times \nabla v\right)_z + D + S^2 = 0, \quad (20.17)$$

where the cross product represents vector multiplication of the horizontal gradients of α and v.

As a sample calculation, we consider a circular vortex centered at an origin $r = 0$ with constant angular velocity ω, like that for rigid body rotation. In this case, all deformation components vanish (e.g., J. SERRIN [4]), thus D dissappears. Here $v = r\omega$,

$$\frac{\partial}{\partial s} = \frac{\partial}{r\partial \alpha}, \qquad \frac{\partial}{\partial n} = -\frac{\partial}{\partial r},$$

and equation (20.15) immediately results in:

$$-\omega^2 + S^2 = 0, \quad (20.18)$$

that is, the spin of the individual fluid particles corresponds to the angular velocity of the entire circular vortex.

It is still noted, that the equations (20.15), (20.16) and (20.17) apply to arbitrary fluids, as well as to stationary and nonstationary two dimensional flows, according to the purely kinetic nature of the generally accepted compatibility determinant.

References

1. ERTEL, H.: Eine Relation zwischen kinematischen Parametern horizontaler Strömungsfelder in der Atmosphäre. *Időjárás*, Budapest, 74 (1970), 98.

2. FORTAK, H.: *Acta Hydrophysica*, Berlin XIV (1969), 57.

3. OLBERG, M.: *Acta Hydrophysica*, Berlin XIV (1969), 181.

4. SERRIN, J.: Mathematical Principles of Classical Fluid Mechanics. Handbuch der Physik, Bd. VIII/1. Berlin/Göttingen/Heidelberg 1959, 139.

5. SUTTON, O. G.: Micrometeorology. London 1953, 23.

6. WEIZEL, W.: Physikalische Formelsammlung. Erster Band. Mannheim 1962, 48.

21 ERTEL (1970c): Transformation of the differential form of the Weber hydrodynamic equations in relation to the Earth's rotation

Summary

We derive a system of equations that originate from the differential form of the Weber formulas of hydrodynamics, including the effects of the Earth's rotation. With this extension the Weber formulas are of importance for meteorology and hydrography. The direct relationship of the extended Weber formulas with the circulation theorem of V. Bjerknes is easily deduced, as is the variational principle of atmospheric

dynamics of H. Ertel.

1. Symbols

In the notation of vector and tensor analysis, including the summation convention, the hydrodynamic momentum equations for autobarotropic flows, relative to an orthogonal cartesian coordinate system x_j ($j = 1, 2, 3$) fixed with respect to the rotating Earth, can be written

$$\frac{dv_j}{dt} - 2\omega_{jk}\, v_k = -\frac{\partial W}{\partial x_j}, \qquad (21.1)$$

where d/dt = individual (material) time derivative, $v_j = dx_j/dt$ = velocity components, $\omega_{jk} = -\omega_{kj}$ = Coriolis parameters ($j, k = 1, 2, 3$) for the Earth's rotation, W = potential energy = $V + \int \alpha dp$, V = potential for the force of gravity, and $\int \alpha dp$ = autobarotropic function of the specific volume $\alpha = \alpha(p)$. The origin $x_j = 0$ of the coordinate system is any point on the Earth's axis.

2. Differential form of the Weber equations

The Weber equations of hydrodynamics use the time-independent Lagrangian coordinates a_i ($i = 1, 2, 3$) as labels of individual particles; they are obtained from the Euler equations (21.1) through multiplication by $\partial x_j/\partial a_i$ and summation over the index j, resulting in the Lagrangian system

$$\frac{dv_j}{dt}\frac{\partial x_j}{\partial a_i} - 2\omega_{jk}\, v_k \frac{\partial x_j}{\partial a_i} = -\frac{\partial W}{\partial a_i}. \qquad (21.2)$$

Because

$$\frac{dv_j}{dt}\frac{\partial x_j}{\partial a_i} = \frac{d}{dt}\left(v_j \frac{\partial x_j}{\partial a_i}\right) - \frac{\partial}{\partial a_i}\left(\frac{v^2}{2}\right), \qquad (21.3)$$

where $v^2 = v_j v_j$, we can write (21.1) as

$$\frac{d}{dt}\left(v_j \frac{\partial x_j}{\partial a_i}\right) - 2\omega_{jk}\, v_k \frac{\partial x_j}{\partial a_i} = \frac{\partial \Omega}{\partial a_i}, \qquad (21.4)$$

where Ω is the difference between kinetic and potential energy:

$$\Omega = \tfrac{1}{2}v^2 - W, \qquad (21.5)$$

which is the (Helmholtz) kinetic potential.

In the absense of rotation ($\omega_{jk} = 0$), V (in W and/or Ω) assumes the simple meaning of the gravitational potential. By time integration of the formulas (21.4), together with $\omega_{jk} = 0$, the Weber formulas would result (Ref.: [3]–[10]). Thus, the formulas (21.4) can be regarded as the differential form of the Weber formulas with the inclusion of the Earth's rotation.

3. Transformation of the differential form of the Weber formulas with inclusion of the Coriolis terms

We now try to transform the formulas (21.4) in such a manner that the Coriolis terms, $-2\omega_{jk}v_k(\partial x_j/\partial a_i) = -2\omega_{jk}(dx_k/dt)(\partial x_j/\partial a_i)$, can be split into two parts involving the operators d/dt and $\partial/\partial a_i$. For this purpose the following identity is useful:

$$2\omega_{jk}\, v_k \frac{\partial x_j}{\partial a_i} = \frac{\partial}{\partial a_i}\left(\omega_{jk}\, x_j \frac{dx_k}{dt}\right) \\ - \frac{d}{dt}\left(\omega_{jk}\, x_j \frac{\partial x_k}{\partial a_i}\right). \qquad (21.6)$$

The correctness of this identity can be easily verified by evaluation of the derivatives with consideration of the commutative property of the operators d/dt and $\partial/\partial a_i$. The first term of the right side of (21.6) can be expanded as

$$\frac{\partial}{\partial a_i}\left(\omega_{jk}\, x_j \frac{dx_k}{dt}\right) = \omega_{jk}\frac{\partial x_j}{\partial a_i}\frac{dx_k}{dt} \\ + \omega_{jk}\, x_j \frac{d}{dt}\frac{\partial x_k}{\partial a_i}, \qquad (21.7)$$

and the second term as

$$-\frac{d}{dt}\left(\omega_{jk}\, x_j \frac{\partial x_k}{\partial a_i}\right) = -\omega_{jk}\frac{dx_j}{dt}\frac{\partial x_k}{\partial a_i} \\ - \omega_{jk}\, x_j \frac{d}{dt}\frac{\partial x_k}{\partial a_i}. \qquad (21.8)$$

The addition of (21.7) and (21.8), with consideration of

$$\omega_{jk}\frac{\partial x_j}{\partial a_i}\frac{dx_k}{dt} - \omega_{jk}\frac{dx_j}{dt}\frac{\partial x_k}{\partial a_i} \\ = \omega_{jk}\, v_k \frac{\partial x_j}{\partial a_i} - \omega_{kj}\, v_k \frac{\partial x_j}{\partial a_i} \qquad (21.9)$$

and the antisymmetry of the Coriolis components ω_{kj}, verifies the identity (21.6).

The substitution of the identity (21.6) in (21.4) transforms the differential form of the Weber formulas with the Coriolis terms into the system

$$\boxed{\begin{aligned} \frac{d}{dt}&\left(v_j \frac{\partial x_j}{\partial a_i} + \omega_{jk}\, x_j \frac{\partial x_k}{\partial a_i}\right) \\ &= \frac{\partial}{\partial a_i}\left(\Omega + \omega_{jk}\, x_j \frac{dx_k}{dt}\right), \end{aligned}} \qquad (21.10)$$

which is very suitable for applications to meteorology and hydrography, as the following examples show.

4. Applications

54

The multiplication of the formulas (21.10) with the differentials da_i of the time-independent Lagrangian labeling coordinates a_i and summation over the index i results in

$$da_i \frac{d}{dt}\left(v_j \frac{\partial x_j}{\partial a_i} + \omega_{jk} x_j \frac{\partial x_k}{\partial a_i}\right)$$
$$= d\left(\Omega + \omega_{jk} x_j \frac{dx_k}{dt}\right), \tag{21.11}$$

from which, by integration over a closed material curve C, it follows that

$$\frac{d}{dt}\int_C (v_j dx_j + \omega_{jk} x_j \, dx_k) = 0. \tag{21.12}$$

This is the conservation theorem for the absolute circulation in autobarotropic currents (Ref.: [1]), where $\int_C v_j \, dx_j$ is the circulation relative to the Earth and the second integral is twice the absolute value of the Earth's rotation vector (ω) multiplied by the projection F of the closed curve C on the equatorial plane (or on a plane parallel to the equatorial plane). In other words,

$$\int_C \omega_{jk} x_j dx_k = \int_C \omega_{jk} \tfrac{1}{2}(x_j dx_k - x_k dx_j)$$
$$= 2\omega F, \tag{21.13}$$

where $\omega_1 = \omega_{23} = -\omega_{32}$, $\omega_2 = \omega_{31} = -\omega_{13}$, and $\omega_3 = \omega_{12} = -\omega_{21}$ are the three components $(\omega_1, \omega_2, \omega_3)$ of the Earth's rotation vector.

In just as simple a fashion one can obtain from (21.10) the variational principle of atmospheric dynamics (Ref.: [2]). Multiplying (21.10) by the variations δa_i of the time-independent Lagrangian labeling coordinates and then summing over the index i, we obtain the formulas

$$\frac{d}{dt}(v_j \, \delta x_j + \omega_{jk} x_j \, \delta x_k) = \delta a_i \frac{\partial}{\partial a_i}\left(\Omega + \omega_{jk} x_j \frac{dx_k}{dt}\right)$$
$$= \delta(\Omega + \omega_{jk} x_j v_k). \tag{21.14}$$

Integrating over the time interval $t_2 - t_1$, and noting that the variations $\delta x_j = (\partial x_j/\partial a_i)\delta a_i$ vanish at the end points t_1 and t_2 for all indices j, the variational principle of atmospheric dynamics in the form

$$0 = \delta \int_{t_1}^{t_2} L \, dt \tag{21.15}$$

results where the (generalized) Lagrange function

$$L = \Omega + \omega_{jk} x_j v_k, \tag{21.16}$$

contains the kinetic potential Ω and the parallel component of the spin of a particle around the coordinate origin (which is on the Earth's axis), since the term $\omega_{jk}x_j v_k$ represents simply the dot product of the Earth's rotation vector with the spin vector.

References

[1] BJERKNES, V., BJERKNES, J., SOLBERG, H., u. T. BERGERON, Physikalische Hydrodynamik mit Anwendung auf die dynamische Meteorologie, Berlin 1933, 147.

[2] ERTEL, H., Methoden und Probleme der dynamischen Meteorologie, Berlin 1938, 41; amerikanische Ausgabe: Ann Arbor (Mich.) 1943.

[3] LAMB, H., Hydrodynamics, 6th (rev.) ed., 1st amer. ed. New York 1945, 14.

[4] LICHTENSTEIN, L., Grundlagen der Hydromechanik, Berlin 1929, 408.

[5] LONG, R. R., Mechanics of Solids and Fluids, Englewood Cliffs (N.J.) 1961, 129.

[6] OSWATITSCH, K., Physikalische Grundlagen der Strömungslehre, in: Handbuch der Physik, Bd. VIII/1, Berlin/Göttingen/Heidelberg 1959, 32.

[7] SCHAEFER, C., Einführung in die theoretische Physik, 1. Bd., 4. (durchges.) Aufl. Berlin 1944, 794.

[8] SERRIN, J., Mathematical Principles of Classical Fluid Mechanics, in: Encyclopedia of Physics, Vol. VIII/1, Berlin/Göttingen/Heidelberg 1959, 169.

[9] WEBER, H., Die partiellen Differentialgleichungen der mathematischen Physik, nach Riemann's Vorlesungen bearbeitet, 2. Bd., 6. Aufl. Braunschweig 1919, 419.

[10] WEIZEL, W., Physikalische Formelsammlung. 1. Bd., Mannheim 1962, 89.