escape per second; and 424 kilogram-metres being required to raise 1 kilog. of water 1° Cent., $424 \times t$ kilogram-metres are required to raise 1 kilog. by $t°$; it follows, therefore, that $424 \times t° \times 1000\,Q$ is the number of kilogram-metres necessary to raise the water which escapes per 1″ by $t°$;

$$\therefore\ 424 \times t° \times 1000\,Q = U_3 + U_4,$$

or

$$t° = \frac{U_3 + U_4}{424000\,Q}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

in which expression Q, U_3, and U_4 are given by equations (31) (42), and (43).

[To be continued.]

XLVI. *Hydrokinetic Solutions and Observations.* *By* Sir William Thomson, *F.R.S.**

Part I. *On the Motion of Free Solids through a Liquid.*

THIS paper commences with the following extract from the author's private journal, of date January 6, 1858:—

"Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}$ be rectangular components of an impulsive force and an impulsive couple applied to a solid of invariable shape, with or without inertia of its own, in a perfect liquid, and let $u, v, w, \varpi, \rho, \sigma$ be the components of linear and angular velocity generated. Then, if the *vis viva*† (twice the mechanical value) of the whole motion be, as it cannot but be, given by the expression

$$Q = [u, u]u^2 + [v, v]v^2 + \ldots + 2[v, u]vu + 2[w, u]wu + \ldots$$
$$+ 2[\varpi, u]\varpi u + \ldots,$$

where $= [u, u]$, $[v, v]$, &c. denote 21 constant coefficients determinable by transcendental analysis from the form of the surface of the solid, probably involving only elliptic transcendentals when the surface is ellipsoidal: involving, of course, the moments of inertia of the solid itself: we must have

$$[u,u]u + [v,u]v + [w,u]w + [\varpi,u]\varpi + [\rho,u]\rho + [\sigma,u]\sigma = \mathfrak{X}, \&c.,$$
$$[u,\varpi]u + [v,\varpi]v + [w,\varpi]w + [\varpi,\varpi]\varpi + [\rho,\varpi]\rho + [\sigma,\varpi]\sigma = \mathfrak{L}, \&c.$$

* Communicated by the Author. Parts I. and II. from the Proceedings of the Royal Society of Edinburgh for 1870–71. Parts III. and IV. from letters to Professor Tait of August 1871. Part V. appended to this communication September 1871.

† Henceforth T, instead of $\frac{1}{2}Q$, is used to denote the "mechanical value," or, as it is now called, the "kinetic energy" of the motion.

If now a continuous force X, Y, Z, and a continuous couple L, M, N, referred to axes fixed in the body, is applied, and if $\mathfrak{X}, \ldots$, &c. denote the impulsive force and couple capable of generating from rest the motion $u, v, w, \varpi, \rho, \sigma$, which exists in reality at any time t; or, merely mathematically, if $\mathfrak{X}$ &c. denote for brevity the preceding linear functions of the components of motion, the equations of motion are as follow :—

$$\left.\begin{array}{l} \dfrac{d\mathfrak{X}}{dt} - \mathfrak{Y}\sigma + \mathfrak{Z}\rho = \mathrm{X}, \quad \dfrac{d\mathfrak{Y}}{dt} = \&\mathrm{c.}, \\ \dfrac{d\mathfrak{L}}{dt} - \mathfrak{Y}w + \mathfrak{Z}v - \mathfrak{M}\sigma + \mathfrak{N}\rho = \mathrm{L}, \\ \dfrac{d\mathfrak{M}}{dt} - \mathfrak{Z}u + \mathfrak{X}w - \mathfrak{N}\varpi + \mathfrak{L}\sigma = \mathrm{M}, \\ \dfrac{d\mathfrak{N}}{dt} - \mathfrak{X}v + \mathfrak{Y}u - \mathfrak{L}\rho + \mathfrak{M}\varpi = \mathrm{N}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Three first integrals, when

$$\mathrm{X}=0, \quad \mathrm{Y}=0, \quad \mathrm{Z}=0, \quad \mathrm{L}=0, \quad \mathrm{M}=0, \quad \mathrm{N}=0, \quad . \quad (2)$$

must of course be, and obviously are,

$$\mathfrak{X}^2 + \mathfrak{Y}^2 + \mathfrak{Z}^2 = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

resultant momentum constant;

$$\mathfrak{L}\mathfrak{X} + \mathfrak{M}\mathfrak{Y} + \mathfrak{N}\mathfrak{Z} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

resultant of moment of momentum constant; and

$$u\mathfrak{X} + v\mathfrak{Y} + w\mathfrak{Z} + \varpi\mathfrak{L} + \rho\mathfrak{M} + \sigma\mathfrak{N} = \mathrm{Q}." \quad . \quad . \quad (5)$$

These equations were communicated in a letter to Professor Stokes, of date (probably January) 1858, and they were referred to by Professor Rankine, in his first paper on Stream-lines, communicated to the Royal Society of London*, July 1863.

They are now communicated to the Royal Society of Edinburgh, and the following proof is added :—

Let P be any point fixed relatively to the body; and at time t, let its coordinates relatively to axes OX, OY, OZ, fixed in space, be x, y, z. Let PA, PB, PC be three rectangular axes fixed re-

* These equations will be very conveniently called the Eulerian equations of the motion. They correspond precisely to Euler's equations for the rotation of a rigid body, and include them as a particular case. As Euler seems to have been the first to give equations of motion in terms of coordinate components of velocity and force referred to lines fixed relatively to the moving body, it will be not only convenient, but just, to designate as "Eulerian equations" any equations of motion in which the lines of reference, whether for position, or velocity, or moment of momentum, or force, or couple, move with the body, or the bodies, whose motion is the subject.

latively to the body, and (A, X), (A, Y),... the cosines of the nine inclinations of these axes to the fixed axes OX, OY, OZ.

Let the components of the "impulse"* or generalized momentum parallel to the fixed axes be ξ, η, ζ, and its moments round the same axes λ, μ, ν; so that if X, Y, Z be components of force acting on the solid, in line through P, and L, M, N components of couple, we have

$$\left.\begin{aligned} &\frac{d\xi}{dt}=\mathrm{X}, \quad \frac{d\eta}{dt}=\mathrm{Y}, \quad \frac{d\zeta}{dt}=\mathrm{Z}, \\ &\frac{d\lambda}{dt}=\mathrm{L}+\mathrm{Z}y-\mathrm{Y}z, \quad \frac{d\mu}{dt}=\mathrm{M}+\mathrm{X}z-\mathrm{Z}x, \quad \frac{d\nu}{dt}=\mathrm{N}+\mathrm{Y}x-\mathrm{X}y. \end{aligned}\right\} \quad (6)$$

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ and $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ be the components and moments of the impulse relatively to the axes PA, PB, PC moving with the body. We have

$$\left.\begin{aligned} &\xi=\mathfrak{X}(\mathrm{A},\mathrm{X})+\mathfrak{Y}(\mathrm{B},\mathrm{X})+\mathfrak{Z}(\mathrm{C},\mathrm{X}), \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\lambda=\mathfrak{L}(\mathrm{A},\mathrm{X})+\mathfrak{M}(\mathrm{B},\mathrm{X})+\mathfrak{N}(\mathrm{C},\mathrm{X})+\mathfrak{Z}y-\mathfrak{Y}z, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}\right\} \quad . \quad (7)$$

Now let the fixed axes OX, OY, OZ be chosen coincident with the position at time t of the moving axes PA, PB, PC: we shall consequently have

$$\left.\begin{aligned} &x=0, \quad y=0, \quad z=0, \\ &\frac{dx}{dt}=u, \quad \frac{dy}{dt}=v, \quad \frac{dz}{dt}=w, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left.\begin{aligned} &(\mathrm{A},\mathrm{X})=(\mathrm{B},\mathrm{Y})=(\mathrm{C},\mathrm{Z})=1, \\ &(\mathrm{A},\mathrm{Y})=(\mathrm{A},\mathrm{Z})=(\mathrm{B},\mathrm{X})=(\mathrm{B},\mathrm{Z})=(\mathrm{C},\mathrm{X})=(\mathrm{C},\mathrm{Y})=0, \\ &\frac{d(\mathrm{A},\mathrm{Y})}{dt}=\sigma, \quad \frac{d(\mathrm{B},\mathrm{X})}{dt}=-\sigma, \quad \frac{d(\mathrm{C},\mathrm{X})}{dt}=\rho, \\ &\frac{d(\mathrm{A},\mathrm{Z})}{dt}=-\rho, \quad \frac{d(\mathrm{B},\mathrm{Z})}{dt}=\varpi, \quad \frac{d(\mathrm{C},\mathrm{Y})}{dt}=-\varpi. \end{aligned}\right\} \quad (9)$$

Using (7), (8), and (9) in (6), we find (1).

One chief object of this investigation was to illustrate dynamical effects of heliçoidal property (that is, right or left-handed asymmetry). The case of complete isotropy, with heliçoidal quality, is that in which the coefficients in the quadratic expression for T fulfil the following conditions.

* See "Vortex Motion," § 6, Trans. Roy. Soc. Edin. (1868).

$$\left.\begin{array}{l}[u,u]=[v,v]=[w,w] \quad \text{(let } m \text{ be their common value),} \\ [\varpi,\varpi]=[\rho,\rho]=[\sigma,\sigma] \quad ,, \; n \quad ,, \quad ,, \quad ,, \\ [u,\varpi]=[v,\rho]=[w,\sigma] \quad ,, \; h \quad ,, \quad ,, \quad ,, \\ [v,w]=[w,u]=[u,v]=0; \;\; [\rho,\sigma]=[\sigma,\varpi]=[\varpi,\rho]=0, \\ \text{and} \\ [u,\rho]=[u,\sigma]=[v,\sigma]=[v,\varpi]=[w,\varpi]=[w,\rho]=0\,; \end{array}\right\} (10)$$

so that the formula for T is

$$T=\tfrac{1}{2}\{m(u^2+v^2+w^2)+n(\varpi^2+\rho^2+\sigma^2)+2h(u\varpi+v\rho+w\sigma)\} \quad (11)$$

For this case, therefore, the Eulerian equations (1) become

$$\left.\begin{array}{l}\dfrac{d(mu+h\varpi)}{dt}-m(v\sigma-w\rho)=X, \text{ \&c.,} \\ \text{and} \\ \dfrac{d(n\varpi+hu)}{dt}=L, \text{ \&c.} \\ \text{[Memorandum:—Lines of reference fixed relatively} \\ \text{to the body.]}\end{array}\right\} . \quad (11)$$

But inasmuch as (11) remains unchanged when the lines of reference are altered to any other three lines at right angles to one another through P, it is easily shown directly from (6), (7), and (9) that if, altering the notation, we take u, v, w to denote the components of the velocity of P parallel to three fixed rectangular lines, and ϖ, ρ, σ the components of the body's angular velocity round these lines, we have

$$\left.\begin{array}{l}\dfrac{d(mu+h\varpi)}{dt}=X, \text{ \&c.,} \\ \text{and} \\ \dfrac{d(n\varpi+hu)}{dt}+h(\sigma v-\rho w)=L, \text{ \&c.} \\ \text{[Memorandum:—Lines of reference fixed in space],}\end{array}\right\} . \quad (12)$$

which are more convenient than the Eulerian equations.

The integration of these equations, when neither force nor couple acts on the body (X = 0 &c., L = 0 &c.), presents no difficulty; but its result is readily seen from § 21 ("Vortex Motion") to be that, when the impulse is both translatory and rotational, the point P, round which the body is isotropic, moves uniformly in a circle or spiral so as to keep at a constant distance from the "axis of the impulse," and that the components of angular velocity round the three fixed rectangular axes are constant.

An isotropic heliçoid may be made by attaching projecting vanes to the surface of a globe in proper positions; for instance, cutting at 45° each, at the middles of the twelve quadrants of any three great circles dividing the globe into eight quadrantal triangles. By making the globe and the vanes of light paper, a body is obtained rigid enough and light enough to illustrate by its motions through air the motions of an isotropic heliçoid through an incompressible liquid. But curious phenomena, not deducible from the present investigation, will, no doubt, on account of viscosity, be observed.

Part II.

Still considering only one moveable rigid body, infinitely remote from disturbance of other rigid bodies, fixed or moveable, let there be an aperture or apertures through it, and let there be irrotational circulation or circulations (§ 60, "Vortex Motion") through them. Let ξ, η, ζ be the components of the "impulse" at time t, parallel to three fixed axes, and λ, μ, ν its moments round these axes, as above, with all notation the same, we still have (§ 26, "Vortex Motion")

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \mathrm{X}, \text{ \&c.}, \\ \frac{d\lambda}{dt} &= \mathrm{L} + \mathrm{Z}y - \mathrm{Y}z, \text{ \&c.} \end{aligned}\right\} \quad . \quad (6) \text{ (repeated).}$$

But, instead of for T a quadratic function of the components of velocity as before, we now have

$$\mathrm{T} = \mathrm{E} + \tfrac{1}{2}\{[u, u]u^2 + \ldots + 2[u, v]uv + \ldots\}, \quad . \quad . \quad (13)$$

where E is the kinetic energy of the fluid motion when the solid is at rest, and $\frac{1}{2}\{[u, u]u^2 + \ldots\}$ is the same quadratic as before. The coefficients $[u, u]$, $[u, v]$, &c. are determinable by a transcendental analysis, of which the character is not at all influenced by the circumstance of there being apertures in the solid. And instead of $\xi = \frac{d\mathrm{T}}{du}$, &c., as above, we now have

$$\left.\begin{aligned} &\xi = \frac{d\mathrm{T}}{du} + \mathrm{I}l, \quad \eta = \frac{d\mathrm{T}}{dv} + \mathrm{I}m, \quad \zeta = \frac{d\mathrm{T}}{dw} + \mathrm{I}n, \\ &\lambda = \frac{d\mathrm{T}}{d\varpi} + \mathrm{I}(ny - mz) + \mathrm{G}l, \qquad \mu = \text{\&c.}, \ \nu = \text{\&c.}, \end{aligned}\right\} \quad . \quad (14)$$

where I denotes the resultant "impulse" of the cyclic motion when the solid is at rest, l, m, n its direction-cosines, G its "rotational moment" (§ 6, "Vortex Motion"), and x, y, z the coordinates of any point in its "resultant axis." These, (14) with (13), used in (6) give the equations of the solid's motion referred to fixed rectangular axes. They have the inconvenience of the coefficients $[u, u]$, $[u, v]$, &c. being functions of the angular coordinates of the solid. The Eulerian equations (free from this inconvenience) are readily found on precisely the same plan as that adopted above for the old case of no cyclic motion in the fluid.

The formulæ for the case in which the ring is circular, has no rotation round its axis, and is not acted on by applied forces, though, of course, easily deduced from the general equations (14), (13), (6), are more readily got by direct application of first principles. Let P be such a point in the axis of the ring, and $\mathfrak{C}$, A, B such constants that $\frac{1}{2}(\mathfrak{C}\omega^2+\mathrm{A}u^2+\mathrm{B}v^2)$ is the kinetic energy due to rotational velocity ω round D, any diameter through P, and translational velocities u along the axis and v perpendicular to it. The impulse of this motion, together with the supposed cyclic motion, is therefore compounded of

$$\text{momentum in lines through P}\begin{cases}\mathrm{A}u+\mathrm{I} \text{ along the axis,}\\ \mathrm{B}v \text{ perpendicular to axis,}\end{cases}$$

and moment of momentum $\mathfrak{C}\omega$ round the diameter D.

Hence if OX be the axis of resultant momentum, (x, y) the coordinates of P relatively to fixed axes OX, OY; θ the inclination of the axis of the ring to O; and ξ the constant value of the resultant momentum, we have

$$\left.\begin{aligned}&\xi\cos\theta=\mathrm{A}u+\mathrm{I}; \quad -\xi\sin\theta=\mathrm{B}v; \quad \xi y=\mathfrak{C}\omega;\\ \text{and}\quad &\dot{x}=u\cos\theta-v\sin\theta; \quad \dot{y}=u\sin\theta+v\cos\theta; \quad \dot{\theta}=\omega.\end{aligned}\right\} \quad (15)$$

Hence for θ we have the differential equation

$$\mathrm{A}\mathfrak{C}\frac{d^2\theta}{dt^2}+\xi\left[\mathrm{I}\sin\theta+\frac{\mathrm{A}-\mathrm{B}}{2\mathrm{B}}\xi\sin 2\theta\right]=0, \quad (16)$$

which shows that the ring oscillates rotationally according to the law of a horizontal magnetic needle carrying a bar of soft iron rigidly attached to it parallel to its magnetic axis.

When θ is and remains infinitely small, $\dot{\theta}$, y, and $\dot{y}$ are each infinitely small, $\dot{x}$ remains infinitely nearly constant, and the ring experiences an oscillatory motion in period

$$2\pi\sqrt{\frac{\mathrm{B}\mathfrak{C}}{[\mathrm{I}+(\mathrm{A}-\mathrm{B})\dot{x}](\mathrm{I}+\mathrm{A}\dot{x})}},$$

compounded of translation along OY and rotation round the diameter D. This result is curiously comparable with the well-known gyroscopic vibrations.

Part III. *The Influence of Wind on Waves in water supposed frictionless.* (*Letter to* Professor Tait, *of date August* 16, 1871.)

Taking OX vertically downwards and OY horizontal, let

$$x = h \sin n(y - \alpha t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation of the section of the water by a plane perpendicular to the wave-ridges; and let h (the half wave-height) be infinitely small in comparison with $\frac{2\pi}{n}$ (the wave-length). The x-component of the velocity of the water at the surface is then

$$-n\alpha h \cos n(y - \alpha t)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and this (because h is infinitesimal) must be the value of $\frac{d\phi}{dy}$ for the point $(0, y)$, if ϕ denote the velocity-potential at any point (x, y) of the water. Now because

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0,$$

and ϕ is a periodic function of y, and a function of x which becomes zero when $x = \infty$, it must be of the form

$$\mathrm{P} \cos (ny - e)\epsilon^{-nx},$$

where P and e are independent of x and y. Hence, taking $\frac{d\phi}{dx}$, putting $x = 0$ in it, and equating it to (2), we have

$$-\mathrm{P}n \cos (ny - e) = -n\alpha h \cos (ny - n\alpha t)\,;$$

and therefore $\mathrm{P} = \alpha h$, and $e = n\alpha t$; so that we have

$$\phi = \alpha h \epsilon^{-nx} \cos n\,(y - \alpha t). \quad . \quad . \quad . \quad . \quad (3)$$

This, it is to be remarked, results simply from the assumptions that the water is frictionless, that it has been at rest, and that its surface is moving in the manner specified by (1).

If the air were a frictionless liquid moving irrotationally, with a constant velocity V at heights above the water (that is to say, values of $-x$) considerably exceeding the wave-length, its velocity potential ψ, found on the same principle, would be

$$(\mathrm{V} - \alpha) h \epsilon^{nx} \cos n\,(y - \alpha t) + \mathrm{V}y. \quad . \quad . \quad . \quad (4)$$

Let now q denote the resultant velocity at any point (x, y) of the air. Neglecting infinitesimals of the order $(nh)^2$, we have

$$\tfrac{1}{2}q^2 = \tfrac{1}{2}\mathrm{V}^2 - \mathrm{V}(\mathrm{V} - \alpha) nh\epsilon^{nx} \sin n(y - \alpha t). \quad . \quad . \quad (5)$$

Now, if p denote the pressure at any point (x, y) in the air, and σ the density of the air, we have by the general equation for pressure in an irrotationally moving fluid,

$$C-p=\sigma\left(\frac{d\psi}{dt}+\tfrac{1}{2}q^2-gx\right). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Using (4) and (5) in this and putting $C=\frac{1}{2}\sigma V^2$, we find

$$-p=\sigma\left\{-nh(V-\alpha)^2\epsilon^{nx}\sin n(y-\alpha t)-gx\right\}. \quad . \quad . \quad (7)$$

Similarly if p' denote the pressure at any point (x, y) of the water, since in this case q^2 is infinitesimal, we have

$$-p'=nh\alpha^2\epsilon^{-nx}\sin n(y-\alpha t)-gx, \quad . \quad . \quad . \quad (8)$$

the density of the water being taken as unity.

Now let T be the cohesive tension of the separating surface of air and water. The curvature of the surface at any point (x, y) given by equation (1), being $\frac{d^2x}{dy^2}$, is equal to

$$-n^2h\sin n(x-\alpha t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence at any point (x, y) fulfilling (1),

$$p-p'=Tn^2h\sin n(x-\alpha t); \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and by (7) and (8), with for x its value by (1) (which, as h is infinitesimal, only affects their last terms), we have

$$p-p'=h\left\{n[\sigma(V-\alpha)^2+\alpha^2]-g(1-\sigma)\right\}\sin n(x-\alpha t). \quad . \quad (11)$$

This, compared with (10), gives

$$n[\sigma(V-\alpha)^2+\alpha^2]-g(1-\sigma)=Tn^2. \quad . \quad . \quad (12)$$

Let

$$w=\sqrt{\frac{g(1-\sigma)+Tn^2}{(1+\sigma)n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which (being the value of α for $V=0$) is the velocity of propagation of waves with no wind, when the wave-length is $\frac{2\pi}{n}$. Then (12) becomes

$$\frac{\alpha^2+\sigma(V-\alpha)^2}{1+\sigma}=w^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which determines α, the velocity of the same waves when there is wind, of velocity V, in the direction of propagation of the waves. Solving the quadratic, we have

$$\alpha=\frac{\sigma V}{1+\sigma}\pm\left\{w^2-\frac{\sigma V^2}{(1+\sigma)^2}\right\}^{\frac{1}{2}}. \quad . \quad . \quad (15)$$

This result leads to the following conclusions:—

(1) When

$$V < w\sqrt{\frac{1+\sigma}{\sigma}},$$

the values of α are positive and negative; that is to say, waves can travel with or against the wind. The positive value is always greater; that is to say, waves travel faster with than against the wind. The velocity of waves travelling *against* the wind is always less than w, the velocity without wind.

(2) When

$$V < 2w,$$

the velocity of waves travelling with the wind is greater than w. When

$$V = 2w,$$

the velocity of the waves *with* the wind is undisturbed by the wind; a result obvious without analysis. When

$$V > 2w,$$

the velocity of waves travelling with the wind is less than the velocity of the same waves without wind.

(3) When

$$V > w\sqrt{\frac{1+\sigma}{\sigma}},$$

waves of such length that w would be their velocity without wind, cannot travel against the wind.

(4) When

$$V > w\frac{1+\sigma}{\sqrt{\sigma}},$$

there cannot be waves of so small length as that for which the undisturbed velocity is w, and the equilibrium of the water is essentially unstable. And (13) shows that the minimum value of w is

$$\sqrt{\frac{2\sqrt{g\mathrm{T}(1-\sigma)}}{1+\sigma}}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Hence the water with a plane level surface is unstable if the velocity of the wind exceeds

$$\sqrt{\frac{2\sqrt{g\mathrm{T}(1-\sigma^2)}}{\sigma}}.$$

W. T.

PART IV. (*Letter to* Professor Tait, *of date August* 23, 1871.)

Defining a ripple as any wave on water whose length $< 2\pi\sqrt{\frac{\mathrm{T}'}{g'}}$*, where

* Which for pure water = 1·7 centim. (see Part V.).

and

$$\left.\begin{aligned} g' &= g\,\frac{1-\sigma}{1+\sigma}, \\ T' &= \frac{T}{1+\sigma} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

($\sigma = \cdot 00122$), you always see an exquisite pattern of ripples in front of any solid cutting the surface of water and moving horizontally at any speed, fast or slow. The ripple-length is the smaller root of the equation

$$\frac{2\pi}{\lambda}\,T' + \frac{\lambda}{2\pi}\,g' = w^2, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where w is the velocity of the solid. The latter may be a sailing-vessel or a row-boat, a pole held vertically and carried horizontally, an ivory pencil-case, a penknife-blade either edge or flat side foremost, or (best) a fishing-line kept approximately vertical by a lead weight hanging down below water, while carried along at about half a mile per hour by a becalmed vessel. The fishing-line shows both roots admirably; ripples in front, and waves of same velocity (λ the greater root of same equation) in rear. If so fortunate as to be becalmed again, I shall try to get a drawing of the whole pattern, showing the transition at the sides from ripples to waves. When the speed with which the fishing-line is dragged is diminished towards the critical velocity

$$\sqrt{2\sqrt{g'T'}},$$

which is the minimum velocity of a wave, being [see Part V. below] for pure water 23 centims. per second (or $\frac{1}{2\cdot 29}$ of a nautical mile per hour), the ripples in front elongate and become less curved, and the waves in rear become shorter, till at the critical velocity waves and ripples seem nearly equal, and with ridges nearly in straight lines perpendicular to the line of motion. (This is observation.) It seems that the critical velocity may be determined with some accuracy by experiment thus [see Part V. below]:—

Remark that the shorter the ripple-length the greater is the velocity of propagation, and that the moving force of the ripple-motion is partly gravity, but chiefly cohesion; and with very short ripple-length it is almost altogether cohesion, *i. e.* the same force as that which makes a dew drop tremble. The least velocity of frictionless air that can raise a ripple on rigorously quiescent frictionless water is [(16) above]

660 centimetres per second

(being $\frac{1+\sigma}{\sqrt{\sigma}}$ × minimum wave-velocity)

$= 12\cdot 8$ nautical miles per hour.

Observation shows the sea to be ruffled by wind of a much smaller velocity than this. Such ruffling, therefore, is due to viscosity of the air.

W. T.

Postscript to Part IV. (*October* 17, 1871).

The influence of viscosity gives rise to a greater pressure on the anterior than on the posterior side of a solid moving uniformly relatively to a fluid. A symmetrical solid, as for example a globe, moving uniformly through a frictionless fluid, experiences augmentation of pressure in front and behind equally; and diminished pressure over an intervening zone. Observation (as for instance in Mr. J. R. Napier's experiments on his "pressure log," for measuring the speed of vessels, and experiments by Joule and myself*, on the pressure at different points of a solid globe exposed to wind) shows that, instead of being increased, the pressure is sometimes actually diminished on the posterior side of a solid moving through a real fluid such as air or water. Wind blowing across ridges and hollows of a fixed solid (such as the furrows of a field) must, because of the viscosity of the air, press with greater force on the slopes facing it than on the sheltered slopes. Hence if a regular series of waves at sea consisted of a solid body moving with the actual velocity of the waves, the wind would do work upon it, or it would do work upon the air, according as the velocity of the wind were greater or less than the velocity of the waves. This case does not afford an exact parallel to the influence of wind on waves, because the surface particles of water do not move forward with the velocity of the waves as those of the furrowed solid do. Still it may be expected that when the velocity of the wind exceeds the velocity of propagation of the waves, there will be a greater pressure on the posterior slopes than on the anterior slopes of the waves; and *vice versâ*, that when the velocity of the waves exceeds the velocity of the wind, or is in the direction opposite to that of the wind, there will be a greater pressure on the anterior than on the posterior slopes of the waves. In the first case the tendency will be to augment the wave, in the second case to diminish it. The question whether a series of waves of a certain height gradually augment with a certain force of wind or gradually subside through the wind not being strong enough to sustain them, cannot be decided offhand. Towards answering it Stokes's investigation of the work against viscosity of water required to maintain a wave†, gives a most important and suggestive instal-

* "Thermal Effects of Fluids in Motion," Royal Society Transactions, 1860; and Phil. Mag. 1860, vol. xx. p. 552.

† Transactions of the Cambridge Philosophical Society, 1851 ("Effect of Internal Friction of Fluids on the Motion of Pendulums," Section V.).

ment. But no theoretical solution, and very little of experimental investigation, can be referred to with respect to the eddyings of the air blowing across the tops of the waves, to which, by its giving rise to greater pressure on the posterior than on the anterior slopes, the influence of the wind in sustaining and maintaining waves is chiefly if not altogether due.

My attention having been called three days ago, by Mr. Froude, to Scott Russell's Report on Waves (British Association, York, 1844), I find in it a remarkable illustration or indication of the leading idea of the theory of the influence of wind on waves, that the velocity of the wind must exceed that of the waves, in the following statement:—"Let him [an observer studying the surface of a sea or large lake, during the successive stages of an increasing wind, from a calm to a storm] begin his observations "in a perfect calm, when the surface of the water is smooth and "reflects like a mirror the images of surrounding objects. This "appearance will not be affected by even a slight motion of the "air, and a velocity of less than half a mile an hour ($8\frac{1}{2}$ in. per "sec.) does not sensibly disturb the smoothness of the reflecting "surface. A gentle zephyr flitting along the surface from point "to point, may be observed to destroy the perfection of the mir-"ror for a moment, and on departing, the surface remains po-"lished as before; if the air have a velocity of about a mile an "hour, the surface of the water becomes less capable of distinct "reflexion, and on observing it in such a condition, it is to be "noticed that the diminution of this reflecting power is owing "to the presence of those minute corrugations of the superficial "film which form waves of the *third order*. These corrugations "produce on the surface of the water an effect very similar to "the effect of those panes of glass which we see corrugated for "the purpose of destroying their transparency, and these corru-"gations at once prevent the eye from distinguishing forms at a "considerable depth, and diminish the perfection of forms re-"flected in the water. To fly-fishers this appearance is well "known as diminishing the facility with which the fish see their "captors. This first stage of disturbance has this distinguishing "circumstance, that the phenomena on the surface cease almost "simultaneously with the intermission of the disturbing cause, "so that a spot which is sheltered from the direct action of the "wind remains smooth, the waves of the third order being inca-"pable of travelling spontaneously to any considerable distance, "except when under the continued action of the original disturb-"ing force. This condition is the indication of present force, "not of that which is past. While it remains it gives that deep "blackness to the water which the sailor is accustomed to regard "as an index of the presence of wind, and often as the forerunner "of more.

"The second condition of wave motion is to be observed when "the velocity of the wind acting on the smooth water has in- "creased to two miles an hour. Small waves then begin to rise "uniformly over the whole surface of the water; these are waves "of the second order, and cover the water with considerable re- "gularity. Capillary waves disappear from the ridges of these "waves, but are to be found sheltered in the hollows between "them, and on the anterior slopes of these waves. The regu- "larity of the distribution of these secondary waves over the "surface is remarkable; they begin with about an inch of am- "plitude, and a couple of inches long; they enlarge as the ve- "locity or duration of the wave increases; by and by contermi- "nal waves unite; the ridges increase, and if the wind increase "the waves become cusped, and are regular waves of the *second* "*order*. They continue enlarging their dimensions; and the "depth to which they produce the agitation increasing simulta- "neously with their magnitude, the surface becomes extensively "covered with waves of nearly uniform magnitude."

The "Capillary waves" or "waves of the third order" referred to by Russell are what I, in ignorance of his observations on this branch of his subject, had called "ripples." The velocity of $8\frac{1}{2}$ inches ($21\frac{1}{2}$ centimetres) per second is precisely the velocity he had chosen (as indicated by his observations) for the velocity of propagation of the straight-ridged waves streaming obliquely from the two sides of the path of a small body moving at speeds of from 12 to 36 inches per second; and it agrees remarkably with my theoretical and experimental determination of the absolute minimum wave-velocity (23 centimetres per second; see Part V.). Russell has not explicitly pointed out that his critical velocity of $8\frac{1}{2}$ inches per second was an absolute minimum velocity of propagation. But the idea of a minimum velocity of waves can scarcely have been far from his mind when he fixed upon $8\frac{1}{2}$ inches per second as the minimum of wind that can sustain ripples. In an article to appear in 'Nature' on the 26th of this month, I have given extracts from Russell's Report (including part of a quotation which he gives from Poncelet and Lesbros in the memoirs of the French Institute) for 1829, showing how far my observations on ripples had been anticipated. I need say no more here than that these anticipations do not include any indication of the dynamical theory which I have given, and that the subject was new to me when Parts III., IV., and V. of the present communication were written.

Part V. *Waves under motive power of Gravity and Cohesion jointly, without wind.*

Leaving the question of wind, consider (13), and introduce notation of (16), (17) in it. It becomes

$$w^2 = \frac{g'}{n} + T'n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This has a minimum value,

when
$$\left.\begin{aligned} w^2 &= 2\sqrt{g'T'}, \\ n &= \sqrt{\frac{g'}{T'}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In applying these formulæ to the case of air and water, we may neglect the difference between g and g', as the value of σ is about $\frac{1}{820}$; and between T and T′, although it is to be remarked that it is T′ rather than T that is ordinarily calculated from experiments on capillary attraction. From experiments of Gay-Lussac's it appears that the value of T′ is about ·074 of a gramme weight per centimetre; that is to say, in terms of the kinetic unit of force founded on the gramme as unit of mass,

$$T' = g \times \cdot073.$$

To make the density of water unity (as that of the lower liquid has been assumed), we must take one centimetre as unit of length. Lastly, with one second as unit of time, we have

$$g = 982;$$

and (18) gives

$$w = \sqrt{982\left(\frac{1}{n} + \cdot074 \times n\right)}$$

for the wave-velocity in centimetres per second, corresponding to wave-length $\frac{2\pi}{n}$. When $\frac{1}{n} = \sqrt{\cdot073} = \cdot27$ (that is, when the wave-length is 1·7 centimetre), the velocity has a minimum value of 23 centimetres per second.

The part of the preceding theory which relates to the effect of cohesion on waves of liquids occurred to me in consequence of having recently observed a set of very short waves advancing steadily, directly in front of a body moving slowly through water, and another set of waves considerably longer following steadily in its wake. The two sets of waves advanced each at the same rate as the moving body; and thus I perceived that there were two different wave-lengths which gave the same velocity of propagation. When the speed of the body's motion through the water was increased, the waves preceding it became shorter, and those in its wake became longer. Close before the cut-water of a vessel moving at a speed of not more than two or three knots* through very smooth water, the surface of the water is marked with an exquisitely fine and regular fringe of ripples, in which several scores of ridges and hollows may be distinguished (and probably counted, with a little practice) in a space extending 20 or 30 centimetres in advance of the solid. Right astern of either a steamer or sailing vessel moving at any speed above four or five knots, waves may generally be seen following the vessel at exactly its own speed, and appearing of such lengths as to verify as nearly as can be judged the ordinary formula

$$l=\frac{2\pi w^2}{g}$$

for the length of waves advancing with velocity w, in deep water. In the well-known theory of such waves, gravity is assumed as the sole origin of the motive forces. When cohesion was thought of at all (as, for instance, by Mr. Froude in his important nautical experiments on models towed through water, or set to oscillate to test qualities with respect to the rolling of ships at sea), it was justly judged to be not sensibly influential in waves exceeding 5 or 10 centimetres in length. Now it becomes apparent that for waves of any length less than 5 or 10 centimetres cohesion contributes sensibly to the motive system, and that, when the length is a small fraction of a centimetre, cohesion is much more influential than gravity as "motive" for the vibrations.

The following extract from part of a letter to Mr. Froude, forming part of a communication to 'Nature' (to appear on the 26th of this month), describes observations for an experimental determination of the minimum velocity of waves in sea-water:—

"About three weeks later, being becalmed in the Sound of "Mull, I had an excellent opportunity, with the assistance of "Professor Helmholtz, and my brother from Belfast, of determining by observation the minimum wave-velocity with some "approach to accuracy. The fishing-line was hung at a distance "of two or three feet from the vessel's side, so as to cut the "water at a point not sensibly disturbed by the motion of the "vessel. The speed was determined by throwing into the sea "pieces of paper previously wetted, and observing their times of "transit across parallel planes, at a distance of 912 centimetres "asunder, fixed relatively to the vessel by marks on the deck "and gunwale. By watching carefully the pattern of ripples "and waves which connected the ripples in front with the waves "in rear, I had seen that it included a set of parallel waves

* The speed "one knot" is a velocity of one nautical mile per hour, or 51·5 centimetres per second.

" slanting off obliquely on each side, and presenting appearances " which proved them to be waves of the critical length and cor- " responding minimum speed of propagation. Hence the com- " ponent velocity of the fishing-line perpendicular to the fronts " of these waves was the true minimum velocity. To measure " it, therefore, all that was necessary was to measure the angle " between the two sets of parallel lines of ridges and hollows " sloping away on the two sides of the wake, and at the same " time to measure the velocity with which the fishing-line was " dragged through the water. The angle was measured by hold- " ing a jointed two-foot rule, with its two branches, as nearly as " could be judged by the eye, parallel to the set of lines of wave " ridges. The angle to which the ruler had to be opened in this " adjustment was the angle sought. By laying it down on paper, " drawing two straight lines by its two edges, and completing a " simple geometrical construction with a length properly intro- " duced to represent the measured velocity of the moving solid, " the required minimum wave-velocity was readily obtained. " Six observations of this kind were made, of which two were " rejected as not satisfactory. The following are the results of " the other four:—

Velocity of moving solid.	Deduced minimum wave-velocity.
51 centimetres per second.	23·0 centimetres per second.
38 " "	23·8 " "
26 " "	23·2 " "
24 " "	22·9 " "
Mean . .	23·22

"The extreme closeness of this result to the theoretical esti- " mate (23 centimetres per second) was, of course, merely a co- " incidence; but it proved that the cohesive force of sea-water " at the temperature (not noted) of the observation cannot be " very different from that which I had estimated from Gay- " Lussac's observations for pure water."

XLVII. *Preliminary Catalogue of the Bright Lines in the Spectrum of the Chromosphere.* *By* C. A. Young, *Ph.D., Professor of Astronomy in Dartmouth College**.

THE following list contains the bright lines which have been observed by the writer in the spectrum of the chromosphere within the past four weeks. It includes, however, only

* From the American Journal of Science and Art for November, communicated in advance by the Author.