

ON FLUID MOTIONS PRODUCED BY DIFFERENCES OF TEMPERATURE AND HUMIDITY.

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Let us consider a fluid at rest under no external forces save gravity and the pressures over its boundaries. We wish to know in what circumstances, if any, the fluid can remain at rest when heat or water vapour is supplied or extracted over closed surfaces within the fluid. The dynamical equations to be satisfied, if the fluid is to be in equilibrium, are the three fundamental equations of hydrostatics, namely:—

$$\left. \begin{aligned} -\frac{1}{\rho} \frac{\partial p}{\partial x} + X &= 0 \\ -\frac{1}{\rho} \frac{\partial p}{\partial y} + Y &= 0 \\ -\frac{1}{\rho} \frac{\partial p}{\partial z} + Z &= 0 \end{aligned} \right\} \quad (1)$$

where p is the pressure, ρ the density, x, y, z rectangular coordinates of position, and X, Y, Z components of force per unit mass. If U be the gravitation potential, so that

$$(X, Y, Z) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)U \quad (2)$$

the equations (1) can be combined into the single equation

$$dp = -\rho dU \quad (3)$$

and since p, ρ and U are all single-valued functions of position this relation is possible only if p and ρ are functions of U alone. Thus the level surfaces must be surfaces of equal pressure and equal density.

Suppose now that the fluid is of uniform composition. Then the density is a single-valued function of pressure and temperature, and the temperature is therefore a function of pressure and density, and will be a single-valued function unless the temperatures within the system include ranges both above and below the temperature of maximum density of the fluid, if it has one. Hence the temperature also is a function of U , and the isothermal surfaces will be level surfaces. So long as the temperature varies continuously from point to point, this will remain true even if parts of the fluid are above and others below the temperature of maximum density. For if the same level surface passed through regions where the fluid had the same density, as it must, but where the temperature was in some places above and in others below that of maximum density, the temperature would have a finite discontinuity at the transition from one to the other; and this is impossible except instantaneously, and thus heat conduction between such regions would increase the density and destroy the equilibrium. In all cases, therefore, the isothermal surfaces in a state of equilibrium must be horizontal.

Now conduction of heat can take place only towards lower temperature; the rate of flow of heat across any element of surface dS in the fluid is $k \frac{\partial V}{\partial n} dS$, where k is the thermal conductivity, V

the temperature, and ∂n the element of a normal to the element. Since the temperature is uniform over the level surfaces, it follows that there is no flow of heat along the level surfaces; all the flow takes place along lines of gravitational force. Imagine now the tubes of force within the system. So long as the temperature of the fluid is steady¹ there must be no accumulation or loss of heat at any point within it; and by integration there must be no accumulation or loss of heat in any tube of force within the fluid. Thus the rate of flow of heat across every section of the same tube of force must be the same.

If a region R within the fluid is enclosed by a solid boundary, so that the fluid is both above it and below it, some tubes of force will pass through the solid, and we wish to know whether the conduction into the solid over the upper surface will just balance the conduction out of it over the lower, or whether there may be an excess in one direction. This can be answered by the following device. The flow of heat outside the region R will not be altered if the solid is supposed removed without disturbing the fluid, and then replaced by fluid of the same composition and with the same distribution of temperature and density maintained over the surface where it meets the original fluid. If, further, the isothermal surfaces within the new fluid are also the level surfaces, all the conditions for the new system to be in a steady state are satisfied. Therefore there can be no accumulation or loss of heat inside any tube of force even when the tube passes through the region R . Thus the flow into R over the upper section of the tube by the boundary of R is equal to the flow out of R over the lower section of the tube by the boundary of R . But the flow across an element of area depends simply on the temperature gradients across that element, and the temperature at every point outside of R is precisely what it was when R was filled by a solid instead of a liquid. Thus when R was filled by a solid the flow into the liquid from it must also have just balanced the flow from the liquid into R . Thus we have the theorem:—

If a fluid is in equilibrium, every level surface within it, or in contact with it, must also be isothermal, and if any solid is wholly surrounded by fluid the total rate of inflow of heat from the solid to the fluid is zero.

For physical applications the theorem may be stated in another way, as follows:—

If a difference of temperature is maintained over any level surface within or in contact with a fluid, or if heat is supplied to or withdrawn from any region within the fluid, the fluid will move, and will continue to move until such difference of temperature or supply or removal of heat ceases.

¹ If the temperature is not steady, that is, if it is changing with the time, expansion or contraction will be taking place in regions within the fluid, and this in itself implies motion.

The argument still holds completely even if the earth's rotation is taken into account. The only modification is that U needs to be increased by the "potential of the centrifugal forces," namely, $\frac{1}{2}\Omega^2 P^2$, where Ω is the earth's angular velocity of rotation and P is the distance from the earth's axis. This device is a familiar one in works on tides and the figure of the earth. The definition of a level surface needs a slight modification to correspond.

2. It can be proved easily that the same theorem holds with respect to the supply of material constituents as well as heat. For the addition of new matter over any surface necessarily implies a mean outward velocity over that surface, and therefore the fluid cannot be at rest. If wherever new fluid is added an equal volume of old fluid is absorbed as, for instance, when a plant absorbs carbon dioxide and returns an equal volume of oxygen, this argument can be evaded. But even in such a case the constituents considered would in general have different densities and different rates of diffusion. Equilibrium would require that the surfaces of equal density should be level surfaces as before, but they need not on this ground alone be isothermal when the composition is not uniform, since the effect of temperature differences on the density might be balanced by that of differences of composition. Permanent equilibrium in this case would, however, be impossible, for heat and each constituent would tend to spread out at different rates; for in general all the diffusion coefficients would be different, and all different from the conductivity of the fluid as a whole. Thus the densities over the level surfaces would cease to be uniform, and any equilibrium of this type could only be transitory. Equilibrium could only be permanent if the theorem proved above for temperature held separately both for temperature and for every material constituent.

These theorems are so general in character that they may be regarded as the fundamental theorems of dynamical meteorology and oceanography.

3. The above theorem, proved by exact methods from the fundamental laws of hydrostatics and heat conduction, disagrees with a principle stated by Sandström,² and apparently proved from thermodynamic principles. This principle is stated as follows (my translation):—"Heating and cooling will therefore always initiate movements of air in the atmosphere, but these will ultimately disappear if the heating takes place at a higher level than the cooling." V. Bjerknes, following Sandström, states the principle as follows³:—"In steady movements maintained by heat in the atmosphere or ocean the source of cold must necessarily lie at a higher level than the source of heat." It has been proved above that if there is heating or cooling at an internal surface of a fluid there must in any case be a permanent motion of the fluid; the relative levels of the sources of heat and cold have nothing to do with the argument.

The argument of Sandström is as follows. In order to maintain a steady motion of a fluid against viscosity a continuous supply of energy is required. This must arise from work done on the

² Sandström, J. W. *Göteborg, Vet. Handl.*, 17 (4), 1916.

³ Leipzig, *Abh. Ges. Wiss.*, 35, 1916, 29.

fluid. If the fluid is in a steady state, its centre of mass has no upward or downward motion, and gravity is doing no work. It follows that the energy arises from work done against fluid pressure in expansion and contraction. But on the whole the fluid is not expanding or contracting; the reduction of volume in some parts of it just balances the increase in others. Thus if the pressure were uniform no work would on the whole be done against it. Therefore work can on the whole be done against pressure only if the expansion is done under greater pressure than the contraction. The places where elements of fluid in their paths become hotter must therefore lie in regions of greater pressure in the fluid than the places where they become colder.

The argument, as it stands, is quite correct. The slip comes in the passage from this stage to the next, where it is inferred that the higher the source of cold lies in comparison with the source of heat the stronger will be the currents produced. This statement may or may not be true, but it does not follow as an immediate consequence of the argument that precedes it without an additional assumption, which is not explicitly stated. This assumption is that the expansion takes place at the place where the high temperature is maintained, and the contraction at the place where the low temperature is maintained. If my theorems as proved in sections 1 and 2 above are combined with the part of Sandström's argument that I have accepted above, it follows indeed that this assumption is not true; for I have shown that permanent movement will result if the source of cold is lower than the source of heat, and Sandström has shown that the expansion must take place at a lower level than the contraction, and therefore the expansion cannot in such a case take place at the same level as the source of heat and the contraction simultaneously at the level of the source of cold.

What I believe actually happens if heat is steadily supplied to a fluid at one region within it and extracted at another is that a warm region is formed in the neighbourhood of the source of heat, and a cold region near the place where heat is removed. Thus a particle is expanding while it is on the way from the cold region to the hot region, and contracting when on the way from the hot region to the cold region. On this interpretation the proper inference to draw from Sandström's argument is that the path from the cold region to the hot one must lie below the return path; an important proposition, and one worth definite proof, but in accordance with ordinary views.

4. In support of the principle that permanent motion can be produced only if the source of heat is at a lower level than the source of cold, Sandström gives some interesting experimental results, which require re-interpretation if this principle is not accepted. Two pipes, one carrying hot water and the other cold, passed through a tank of water on opposite sides. When the cold one was at the higher level a vigorous permanent circulation was produced in the tank. When the hot one was the higher an equally vigorous circulation began, but gradually died down. In the final state the water above the hot pipe was at the temperature of the hot pipe, and that below the cold pipe was at the temperature of the cold pipe. Between these two levels was a transition

layer, wherein the temperature gradually passed from that of the hot pipe to that of the cold one. No motion could be detected in this layer.

Some progress can be made with the theory of this experiment by considering the orders of magnitude of the quantities involved. The system is practically a two-dimensional one, so it will be assumed that there is no motion parallel to the axis of y . We have the equations of motion, the axis of z being now taken vertically upwards.

$$\left. \begin{aligned} \rho \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} - \mu \nabla^2 u - \rho u \frac{\partial u}{\partial x} - \rho w \frac{\partial u}{\partial z} \\ \rho \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} - g\rho + \mu \nabla^2 w - \rho u \frac{\partial w}{\partial x} - \rho w \frac{\partial w}{\partial z} \end{aligned} \right\} \quad (1)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial z}(\rho w) \quad (2)$$

That of thermal expansion is

$$\rho = \rho_0(1 - \alpha V) \quad (3)$$

if V^2 be neglected; and the equation of transfer of heat is

$$\frac{\partial V}{\partial t} = k \nabla^2 V - u \frac{\partial V}{\partial x} - w \frac{\partial V}{\partial z} \quad (4)$$

In these equations all the symbols already used have the meanings previously assigned, while t is the time, u and w the horizontal and vertical components of velocity, μ the true viscosity, g the acceleration due to gravity, ρ_0 the density at the zero of temperature, and α the coefficient of volumetric expansion. V will be used for the variable part of the temperature; in other words, the mean temperature within the apparatus will be taken as the zero of temperature.

So long as the departure of the temperature from uniformity is so small that products of terms depending on the departure can be neglected, equation (4) reduces in a steady state to

$$\nabla^2 V = 0 \quad (5)$$

The isothermal surfaces are therefore closed surfaces, each surrounding one of the two pipes. If then l be of the order of the linear dimensions of the motion, it appears from (1) that the disturbance of the pressure is of order $g\rho_0\alpha V l$, and since the variation of density is gradual that of velocity must be gradual too. Thus $\nabla^2 u$ must be of order u/l^2 , and $\partial p/\partial x$ must be of order $g\rho_0\alpha V$. Hence the equation of horizontal motion shows that u is of order $g\rho_0\alpha V l^2/\mu$. Substituting now in (4) we see that $k(\partial^2 V/\partial x^2)$ is of order kV/l^2 , while $u(\partial V/\partial x)$ is of order $g\rho_0\alpha V^2 l/\mu$. The approximation used in obtaining the equation (5) requires that the second of these shall be small compared with the first. Hence V must be small compared to $\mu k/g\rho_0\alpha l^3$. In Sandström's apparatus l was of the order of 50 cm., while for water we have roughly

$$\mu/\rho_0 = 0.011 \text{ cm.}^2/\text{sec.}; \quad k = 0.0015 \text{ cm.}^2/\text{sec.}; \quad \alpha = 15 \times 10^{-5}/1^\circ\text{C.}$$

With g equal to 981 cm./sec. these show that V must be small compared with 10^{-9} degree. Thus the difference in temperature

between the pipes must be completely insignificant if the terms depending on its square are not to dominate the motion. This result is independent of any assumption that the system has a lapse-rate of temperature capable of leading to instability.

4. 1. In Sandström's experiment the product terms must therefore have had a controlling influence on the motion. In other words, the heat transfer was mainly by convection, for the product terms in (4) express the rate of change of temperature at a place due to the fact that matter there is continually being replaced by new matter from a place where the temperature is somewhat different. Now if the state of the system was steady, the terms on the right of (4) must still give zero on the whole, and this is impossible unless either at least one term of $k \nabla^2 T$ is very large compared with kV/l^2 , or the velocity is very much less than is indicated by the estimate in section 4. The former can only be true if the temperature variations from the values corresponding to a state of equilibrium are practically confined to very thin layers, so that differentiation across the layers introduces factors very large compared with l^{-2} . The confinement of the temperature variations to thin sheets, instead of to regions occupying large fractions of the volume of the apparatus, in its turn reduces the pressure variations and therefore the velocities. It appears, therefore, that in the conditions of the experiment the departures of the temperature from a horizontal stratification must have been extremely small except in very limited regions. It is natural to suppose that the water passing the hot pipe was heated, but probably not quite up to the temperature of the pipe. Thus it would not be able immediately either to rise or to sink, and it would spread out horizontally, becoming thicker as it went on account of interchange of momentum and heat with the fluid above and below by means of viscosity and conduction. A similar sheet, but a cold one, would spread out from the cold pipe. A gentle and probably more diffuse flow would take place from the intermediate layer towards the two pipes to replace the outflowing fluid.

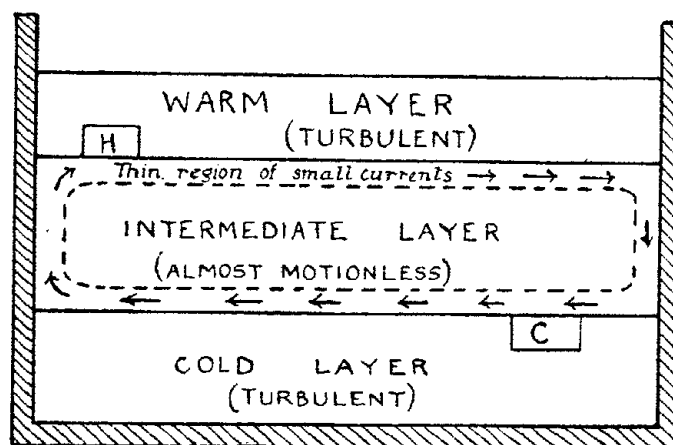


FIG. 1.—Suggested interpretation of Sandström's experiment

Thus the only places where any strong currents were to be expected in the experiment were just on the levels of the two pipes ;

and these are just the places where a current would be practically impossible to observe unless it was of considerable thickness. For heat could travel down through the greater part of the intermediate layer only by conduction, whereas it would travel up from the hot pipe and down from the cold one by turbulence. Thus a certain amount of turbulence was to be expected in the upper and lower layers. This would mask any steady flow where these two layers met the non-turbulent intermediate layer, partly because this flow would be confined to a very thin region, and partly because the injection of any colouring matter there would itself produce a certain amount of turbulence and cause the colouring matter to mix in the neighbouring turbulent layer.

5. In the paper already mentioned Bjerknes states two results obtained as the result of a thermodynamical argument analogous to Sandström's, relating to the movement of water in a closed tube in a vertical plane, two points at opposite sides of the tube being maintained at different, but constant, temperatures. They are, first, that a steady circulation is impossible if the hotter of these points is the higher, and second, that a steady circulation around the tube in either direction is possible if the colder is the higher. If the argument in section 1 is valid the first of these should be untrue; if the result of section 3, that the flow from the cold region to the hot region must take place along a lower path than the return flow, is correct, then the second should also be untrue.

Since the arguments involved are based on general laws involving no approximation, they will, if correct, be applicable to any system so long as it is physically possible. Consider then a closed rectangular tube with its long sides, of length a , vertical, and its short sides, of length b , horizontal. Suppose the viscosity so great or, what is effectively the same thing, the bore of the tube so narrow, that all turbulence is prevented and the motion is slow enough for all square terms to be neglected. Let the temperature at a point A , a distance l below the top side, where l is less than $\frac{1}{2}a$, be maintained at V_1 , and that at B , in the other arm, a distance l above the lowest side, be maintained at V_2 . If then s denote the distance from B , measured counter-clockwise around the tube, the equation of heat conduction in the steady state is

$$\frac{d^2V}{ds^2} = 0 \quad (1)$$

except at A and B , provided the bore is constant. Hence at a point P on the way from B to A the temperature is $V_2 + \frac{V_1 - V_2}{a + b}s$, while that at a point on the way from A to B is

$$V_2 + \frac{V_1 - V_2}{a + b}(2a + 2b - s).$$

The density is, as before, $\rho_0(1 - \alpha V)$.

Now, subject to the conditions that the motion is slow and steady, the acceleration of the fluid is negligible. Thus we have, by considering the motion of the fluid within a short section of the tube,

$$0 = -\frac{\partial p}{\partial s} - n g \rho - F \quad (2)$$

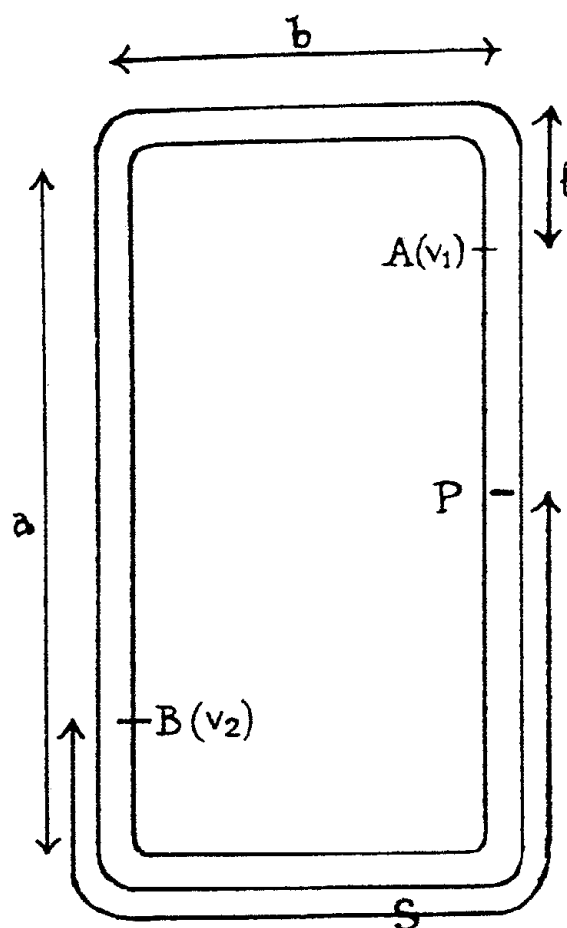


FIG. 2.—Circulation in a closed tube. If B is colder than A the water will move counterclockwise; if B is hotter than A it will move clockwise.

where ng is the component of gravity along the part of the tube considered, and F is the frictional force per unit mass. Hence, integrating right round the tube, we have, since the pressure is a single-valued function,

$$\int F ds = - \int ng \rho_0 (1 - \alpha V) ds = g \rho_0 \alpha \cdot 2l \cdot (V_1 - V_2) \frac{a+b-l}{a+b}. \quad (3)$$

But in viscous motion in a cylindrical tube of radius c the velocity v at a distance r from the axis is of the form

$$v = v_0 \left(1 - \frac{r^2}{c^2} \right) \quad (4)$$

and the frictional force per unit volume is

$$F = -\mu \nabla^2 v = 4\mu v_0 / c^2 \quad (5)$$

Substituting in (3) we have finally

$$\frac{4\mu v_0}{c^2} (a+b)^2 = 2g \rho_0 \alpha l (a+b-l) (V_1 - V_2) \quad (6)$$

It is seen that this result is independent of whether V_1 is greater or less than V_2 , and therefore the fluid will always be capable of a steady motion such that it goes from the cold side to the warm side by way of the bottom, and incapable of one in the reverse sense. This agrees with what was inferred above.

6. The experiment of Sandström and my theory explaining it agree in showing that if regular conduction is the only means of transferring heat from one portion of air to another, and the heat is supplied to the fluid at a higher level than the place where it is removed, the resulting movements of the fluid between these two levels will be narrowly localized and for the most part very slight. If this result were applicable to meteorology it would be one of considerable importance. Actually, however, radiation is capable of redistributing heat through the whole height of the atmosphere, and turbulence can redistribute it throughout the troposphere; and both radiation and turbulence are always present and vastly more important than conduction. It therefore seems improbable that the difference of height between the heated and cooled surfaces has much to do with the strength of the winds produced; any great departure from the ordinary vertical distribution of temperature will be quickly effaced by radiation and turbulence, and thenceforward the winds will be determined by the variations of temperature over horizontal surfaces. Purely local winds, such as land and sea breezes and mountain winds, will be exceptions to this rule, but in the discussion of the great winds, such as the general circulation and the monsoons, it will be legitimate to ignore variations in the height of the land in a first approximation.

The proposition may, however, be of importance in the movements of the ocean, where radiation, and especially dark heat radiation, can penetrate for much shorter distances than in the atmosphere, and turbulence is practically confined to the neighbourhood of the surface. In many oceanic movements conduction may be the chief mode of transfer of heat from one element of water to another, and in such cases the analogy with Sandström's experiment will be very fruitful.

SUMMARY.

It has been shown that the maintenance of a difference of temperature between parts of the same level surface in a fluid will necessarily maintain a permanent motion of the fluid, and that heating or cooling a fluid at an internal boundary will also maintain a permanent movement, however these boundaries may be situated. A corresponding theorem is true for the supply of new constituents instead of heat. This result appears to contradict a theorem given by Sandström and Bjerknes, to the effect that a permanent motion is possible only if the place where the heat is supplied is at a lower level than that where it is removed; but the arguments of these authors involve an unstated assumption, which seems to be untrue. If this assumption is not made, their argument leads to the result that the fluid moving from the lower temperature to the higher is at a lower level than that moving towards the colder regions. These inferences have been verified by working out fully a special case of the motion of a fluid in a closed pipe, heated on one side and cooled on the other. Sandström's experiment, in which no motion was observed in a tank under conditions suited to the production of a circulation, is found to be capable of a dynamical explanation based on the slowness of conduction and the consequent confinement of the currents to narrow regions where they would be very difficult to observe. It appears unlikely that it will often

be possible to proceed by analogy from this experiment to the dynamics of wind, for radiation and turbulence will always redistribute the heat in such a way as to produce general currents; but there may be some applications to ocean currents.

DISCUSSION.

Sir NAPIER SHAW said he would be sorry for an important paper like that of Dr. Jeffreys's to pass without remark. He had already on various occasions exchanged opinions with Dr. Jeffreys upon the subject of vertical circulation in the atmosphere in consequence of the differential warming at places on the same level. He emphasized the difference between laboratory experiments and meteorological sequence, first in respect of the limitation of height to which air could rise in consequence of local heating and the general freedom of local circulation in the atmosphere, and secondly in respect of the possible permanent balancing of pressure differences due to local heating by properly-adjusted air motion. For those reasons he was not inclined to regard differential heating at the same level as being of the same dynamical importance to the atmosphere as differential cooling at different levels.

Col. E. GOLD supposed Dr. Jeffreys meant that there was a limited source of heat. It was possible that in Sandström's experiment, although the sources of cold and of heat were not actual area sources, there was a spreading out so that they tended to set up in the fluid, as it were, area sources and in between the area sources you might merely get conduction without convection. If the area source of cold were above the source of heat, convection must be set up between the two. He saw no reason why convection should be set up between the two if the reverse were the case: although there would be convection above the source of heat and below the source of cold. He hoped Dr. Jeffreys would go on to illustrate the application of his theory in connection with the stratosphere and the troposphere.

Dr. H. JEFFREYS said in reply that he thought Col. Gold was using the term "convection" in a different sense from that in the paper. It was used in the paper to denote the change of an element at a given place due to replacement of the matter at the place by other matter from a place where the element had a different value. This was the sense given in such works as Glazebrook's and Edser's "Heat." If the source of cold were above the source of heat, the system was thoroughly unstable, and the transfer of heat between places was mainly by turbulence, which was only one of the possible types of convection. If, however, the heated source were the higher, there would still be variations of temperature over horizontal surfaces at the levels of the sources, and currents must therefore exist between them. Whether the heat transfer was by convection or conduction was a matter for quantitative test, and it appeared that in the conditions of the experiment considered, convection would be the dominant agency. The motion was not, of course, turbulent in that case, and accordingly the transfer was much slower than when the cold source was the higher. Dr. Jeffreys did not think that the experiment had any meteorological application. The currents must in any case extend as high as the horizontal variations of temperature, and radiation ensured that these extend throughout the whole height of the atmosphere. We were not limited in the atmosphere to what conduction and convection could do for us.