Preface

This document contains problem sets. The problem sets roughly correspond to the material in the corresponding chapter of AOFD (2nd edition).

PROBLEM SET 4

Vorticity and Potential Vorticity

4.1 For the vr vortex, choose a contour of arbitrary shape (e.g., a square) with segments neither parallel nor perpendicular to the radius, and not enclosing the origin. Show explicitly that the circulation around the contour is zero.

4.2 Show that the viscous term in the momentum equation vanishes for a fluid in solid-body rotation.

4.3 ♦ *Vortex stretching and viscosity*
Suppose there is an incompressible swirling flow given in cylindrical coordinates (r, ϕ, z):

$$\boldsymbol{v} = (v_r, v_\phi, v_z) = \left(-\frac{1}{2}\alpha r, v_\phi, \alpha z\right). \tag{P4.1}$$

Show that this satisfies the mass conservation equation. Also, show that vorticity is only non-zero in the vertical direction. Show that the vertical component of the vorticity equation contains only the stretching term and that in a steady state it is

$$-\frac{1}{2}\alpha r \frac{\partial \zeta}{\partial r} = \zeta\alpha + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right). \tag{P4.2}$$

Show that this may be integrated to $\zeta = \zeta_0 \exp(-\alpha r^2/4\nu)$. Thus deduce that there is a rotational core of thickness $r_o = 2\,(\nu/\alpha)^{1/2}$, and that the radial velocity field is given by

$$v_\phi(r) = -\frac{1}{r}\frac{2\nu}{\alpha}\zeta_0 \exp\left[-\frac{\alpha r^2}{4\nu}\right] + \frac{A}{r} \tag{P4.3}$$

where $A = 2\nu\zeta_0/\alpha$. What is the swirling velocity field? (Adapted from ?.)

4.4 Beginning with the three-dimensional vorticity equation in a rotating frame of reference [e.g., (??) or (??)], or otherwise, obtain an expression for the evolution of the vertical and radial coordinate of relative vorticity in Cartesian and spherical coordinates, respectively. Discuss the differences (if any) between the resulting equations and (??). Show carefully how the β-term arises, and in particular that it may be interpreted as arising from tilting term in the vorticity equation.

4.5 ◆ Making use of Kelvin's circulation theorem obtain an expression for the potential vorticity that is conserved following the flow (for an adiabatic and unforced fluid) and that is appropriate for the hydrostatic primitive equations on a spherical planet. Express this in terms of the components of the spherical coordinate system.

4.6 In pressure coordinates for hydrostatic flow on the f-plane, the horizontal momentum equation takes the form

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi \tag{P4.4}$$

On taking the curl of this, there appears to be no baroclinic term. Show that Kelvin's circulation theorem is nevertheless not in general satisfied, even for unforced, adiabatic flow. By appropriately choosing a path on which to evaluate the circulation, obtain an expression for potential vorticity, in this coordinate system, that is conserved following the flow. *Hint:* look at the hydrostatic equation.

4.7 ◆ *Shallow water hurricanes*
Consider the shallow water equations on an f-plane, flat bottom, forced by an imposed steady mass source, and damped by a linear drag:

$$\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u}h) = Q(x, y) \tag{P4.5}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla\boldsymbol{u} + \boldsymbol{f}_0 \times \boldsymbol{u} = -g\nabla h - \alpha\boldsymbol{u}, \tag{P4.6}$$

where α is a positive constant. The term Q represents fluid being added to or removed from the layer by heating.

(a) If we consider the fluid layer to be near the surface of the atmosphere, cumulus convection leads to a mass sink with $Q < 0$. Linearize the equations about a state of rest and solve them to obtain steady solutions for the radial and tangential velocities, and the height field, for the mass sink

$$Q = -Q_0, \quad r < L; \qquad Q = 0 \quad r > L \tag{P4.7}$$

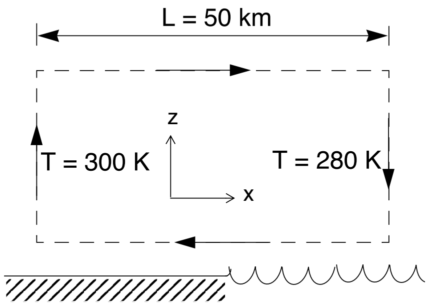
with $Q_0 > 0$. Here $r = \sqrt{x^2 + y^2}$ is radius from the origin. (You may wish to work in plane polar coordinates).

(b) Obtain a scaling estimate for the value of Q_0 at which the linearization will become invalid.

(c) Now consider the *upper* troposphere in a hurricane, where convection leads to an effective mass source so Q is *positive* in some region and zero outside that. Without linearizing the momentum equation, show that in the region of the mass source, in the limit as $\alpha \to 0$, the flow must obtain a state of zero absolute vorticity, $\zeta + f = 0$. You may use the spatial form of the mass source from the previous part of the problem as necessary, although only its sign and not its spatial distribution matters. Explain why this flow violates quasi-geostrophic scaling. (Contributed by Adam Sobel.)

4.8 *Solenoids and sea breezes*

A land–sea temperature contrast of 20 K forces a sea breeze in the surface 'mixed layer' (potential temperature nearly uniform with height), as illustrated schematically. The layer extends through the lowest 10% of the mass of the atmosphere.



(a) In the absence of dissipation and diffusion, at what rate does the circulation change on the material circuit indicated? You may assume the horizontal flow is isobaric, and express your answer in $\mathrm{m\,s^{-1}}$ per hour.

(b) Suppose the sea breeze is equilibrated by a nonlinear surface drag of the form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{V^2}{L_F} \tag{P4.8}$$

with $L_F = (3\,\mathrm{m\,s^{-1}})(3600\,\mathrm{s})$. What is the steady speed of the horizontal wind in the case $L = 50\,\mathrm{km}$?

(c) Suppose that the width of the circulation is determined by a horizontal thermal diffusion of the form

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \kappa_H \frac{\partial^2\theta}{\partial x^2} \tag{P4.9}$$

Provide an estimate of κ_H that is consistent with $L = 50\,\mathrm{km}$. Comment on whether you think the extent of real sea breezes is really determined this way.

PROBLEM SET 5

Geostrophic Theory

5.1 In the derivation of the quasi-geostrophic equations, geostrophic balance leads to the lowest-order velocity being divergence-free — that is, $\nabla_z \cdot \boldsymbol{u}_0 = 0$. It seems that this can also be obtained from the mass conservation equation at lowest order. Is this a coincidence? Suppose that the Coriolis parameter varied, and that the momentum equation yielded $\nabla_2 \cdot \boldsymbol{u}_0 \neq 0$. Would there be an inconsistency?

5.2 ♦ In the planetary-geostrophic approximation, obtain an evolution equation and corresponding inversion conditions that conserves potential vorticity and that are accurate to one higher order in Rossby number than the usual shallow water planetary-geostrophic equations.

5.3 Consider the flat-bottomed shallow water potential vorticity equation in the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{h} = 0 \qquad \text{(P5.1)}$$

(a) Suppose that deviations of the height field are small compared to the mean height field, and that the Rossby number is small (so $|\zeta| \ll f$). Further consider flow on a β-plane such that $f = f_0 + \beta y$ where $|\beta y| \ll f_0$. Show that the evolution equation becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\zeta + \beta y - \frac{f_0 \eta}{H}\right) = 0 \qquad \text{(P5.2)}$$

where $h = H + \eta$ and $|\eta| \ll H$. Using geostrophic balance in the form $f_0 u = -g\partial\eta/\partial y$, $f_0 v = g\partial\eta/\partial x$, obtain an expression for ζ in terms of η.

(b) Linearize (P6.2) about a state of rest, and show that the resulting system supports two-dimensional Rossby waves that are similar to those of the usual two-dimensional barotropic system. Discuss the limits in which the wavelength is much shorter or much longer than the deformation radius.

(c) Linearize (P6.2) about a *geostrophically balanced state* that is translating uniformly eastwards. Note that this means that:

$$u = U + u' \qquad \eta = \eta(y) + \eta',$$

where $\eta(y)$ is in geostrophic balance with U. Obtain an expression for the form of $\eta(y)$.

(d) Obtain the dispersion relation for Rossby waves in this system. Show that their speed is different from that obtained by adding a constant U to the speed of Rossby waves in part ((*b*)), and discuss why this should be so. (That is, why is the problem not *Galilean invariant?*)

5.4 Obtain solutions to the two-layer Rossby wave problem by seeking solutions of the form

$$\psi_1 = \mathrm{Re}\,\widetilde{\psi}_1 \mathrm{e}^{\mathrm{i}(k_x x + k_y y - \omega t)}, \qquad \psi_2 = \mathrm{Re}\,\widetilde{\psi}_2 \mathrm{e}^{\mathrm{i}(k_x x + k_y y - \omega t)}. \tag{P5.3}$$

Substitute (P6.3) directly into (??) to obtain the dispersion relation, and show that the ensuing two roots correspond to the baroclinic and barotropic modes.

5.5 ♦ (Not difficult, but messy.) Obtain the vertical normal modes and the dispersion relationship of the two-layer quasi-geostrophic problem with a free surface, for which the equations of motion linearized about a state of rest are

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 + F_1(\psi_2 - \psi_1)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0 \tag{P5.4a}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + F_2(\psi_1 - \psi_2) - F_{ext}\psi_1\right] + \beta\frac{\partial \psi_2}{\partial x} = 0, \tag{P5.4b}$$

where $F_{ext} = f_0/(gH_2)$.

5.6 Given the baroclinic dispersion relation, $\omega = -\beta k^x/(k^{x2} + k_d^2)$, for what value of k_x is the x-component of the group velocity the largest (i.e., the most positive), and what is the corresponding value of the group velocity?

5.7 Beginning with the vorticity and thermodynamic equations for a two-layer model, obtain an expression for the conversion between available potential energy and kinetic energy in the two-layer model. Show that these expressions are consistent with the conservation of total energy as expressed by (??). Show also that the expression might be considered to be a simple finite-difference approximation to (??).

5.8 ♦ The vertical normal modes are the eigenfunctions of

$$\frac{1}{\widetilde{\rho}}\frac{\partial}{\partial z}\left(\frac{\widetilde{\rho}}{N^2}\frac{\partial \Psi}{\partial z}\right) = -\Gamma\Psi \tag{P5.5}$$

along with boundary conditions on $\Psi(z)$. Obtain numerically the vertical normal modes for some or all of the following profiles, or others of your choice.

(a) $\widetilde{\rho} = 1$, $N^2(z) = 1$, $\Psi_z = 0$ at $z = 0, 1$. (This is profile 'b' of Fig. ??. An analytic solution is possible.)
(b) $\widetilde{\rho} = 1$ and an $N^2(z)$ profile corresponding to a density profile similar to profile 'a' of Fig. ?? (e.g., an exponential), and $\Psi_z = 0$ at $z = 0, 1$.
(c) An isothermal atmosphere, with $\Psi_z = 0$ at $z = 0, 1$. (Similar to (i), except that $\widetilde{\rho}$ varies with height.)
(d) An isothermal atmosphere, now assuming that $\psi \to 0$ as $z \to \infty$.
(e) A fluid with $N^2 = 1$ for $0 < z < 0.5$ and $N^2 = 4$ for $0.5 < z < 1$, with continuous b, and with $\Psi_z = 0$ at $z = 0, 1$.

In the atmospheric cases it is easiest to do the problem first with $\widetilde{\rho} = 1$ (the Boussinesq case) and then extend the problem (and the code) to the compressible case. Then remove the upper boundary to larger and larger values of z.

5.9 ♦ Show that the non-Doppler effect arises using geometric height as the vertical coordinate, using the modified quasi-geostrophic set of ?. In particular, obtain the dispersion relation for stratified quasi-geostrophic flow with a resting basic state. Then obtain the dispersion relation for the equations linearized about a uniformly translating state, paying attention to the lower boundary condition, and note the conditions under which the waves are stationary. Discuss.

5.10 Derive the quasi-geostrophic potential vorticity equation in isopycnal coordinates for a Boussinesq fluid. Show that the isopycnal expression for potential vorticity is approximately equal to the corresponding expression in height coordinates, carefully stating any assumptions that may be necessary to show this.

5.11 *(a)* Obtain the dispersion relationship for Rossby waves in the single-layer quasi-geostrophic potential vorticity equation with linear drag.

(b) Obtain the dispersion relation for Rossby waves in the linearized two-layer potential vorticity equation with linear drag in the lowest layer.

(c) ◆ Obtain the dispersion relation for Rossby waves in the continuously stratified quasi-geostrophic equations, with the effects of linear drag appearing in the thermodynamic equation for the lower boundary condition. That is, the boundary condition at $z = 0$ is $\partial_t(\partial_z\psi)+N^2 w = 0$ where $w = \alpha\zeta$ with α being a constant. You may make the Boussinesq approximation and assume N^2 is constant if you like.

PROBLEM SET 6

Waves

6.1 Consider the flat-bottomed shallow water potential vorticity equation in the form

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