Preface

This document contains problem sets. The problem sets roughly correspond to the material in the corresponding chapter of AOFD (2nd edition).

PROBLEM SET 1

Equations of Motion

1.1 For an infinitesimal volume, informally show that

$$\frac{D}{Dt}(\rho\varphi\Delta V) = \rho\Delta V\frac{D\varphi}{Dt}, \tag{P1.1}$$

where φ is some (differentiable) property of the fluid. Hence informally deduce that

$$\frac{D}{Dt}\int_V \rho\varphi\, dV = \int_V \rho\frac{D\varphi}{Dt}\, dV. \tag{P1.2}$$

1.2 Show that the derivative of an integral is given by

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \varphi(x,t)\, dx = \int_{x_1}^{x_2}\frac{\partial\varphi}{\partial t}\, dx + \frac{dx_2}{dt}\varphi(x_2,t) - \frac{dx_1}{dt}\varphi(x_1,t). \tag{P1.3}$$

By generalizing to three dimensions show that the material derivative of an integral of a fluid property is given by

$$\frac{D}{Dt}\int_V \varphi(\boldsymbol{x},t)\, dV = \int_V \frac{\partial\varphi}{\partial t}\, dV + \int_S \varphi\boldsymbol{v}\cdot d\boldsymbol{S} = \int_V \left[\frac{\partial\varphi}{\partial t} + \nabla\cdot(\boldsymbol{v}\varphi)\right] dV, \tag{P1.4}$$

where the surface integral ($\int_S$) is over the surface bounding the volume V. Hence deduce that

$$\frac{D}{Dt}\int_V \rho\varphi\, dV = \int_V \rho\frac{D\varphi}{Dt}\, dV. \tag{P1.5}$$

1.3 Why is there no diffusion term in the mass continuity equation? Suppose that a fluid contains a binary mixture of dry air and water vapour. Show that the change in mass of a parcel of air due to the diffusion of water vapour is exactly balanced by the diffusion of dry air in the opposite direction.

1.4 By invoking Galilean invariance we can often choose, without loss of generality, the basic state for problems in sound waves to be such that $u_0 \equiv 0$. The perturbation velocity is then certainly larger than the basic state velocity. How can we then justify ignoring the nonlinear term in the perturbation equation, as the term $u'\partial u'/\partial x$ is certainly no smaller than the linear term $u_0\partial u'/\partial x$?

1.5 What amplitude of sound wave is required for the nonlinear terms to become important? Is this achieved at a rock concert (120 dB), or near a jet aircraft that is taking off (160 dB)?

1.6 Using the observed value of molecular diffusion of heat in water, estimate how long it would take for a temperature anomaly to mix from the top of the ocean to the bottom, assuming that molecular diffusion alone is responsible. Comment on whether you think the real ocean has reached equilibrium after the last ice age (which ended about 12 000 years ago).

1.7 Consider the following flow:

$$u = \Gamma z, \qquad v = V \sin[k(x - ct)] ,$$

where Γ, V, k and c are positive constants. (This is similar to the flow in the mid-latitude troposphere — an eastward flow increasing with height, with a transverse wave superimposed.) Suppose that $\Gamma z > c$ for the region of interest. Consider particles located along the $y = 0$ axis at $t = 0$, and compute their position at some later time t. Compare this with the *streamfunction* for the flow at the same time. (*Hint:* show that the meridional particle displacement is $\eta = \psi/(u - c)$, where ψ is the streamfunction and u and c are parameters.)

1.8 ♦ Consider the two-dimensional flow:

$$u = A(y) \sin \omega t, \qquad v = A(y) \cos \omega t.$$

The time-mean of this at a fixed point flow is zero. If A is independent of y, then fluid parcels move clockwise in a circle. What is its radius? If A does depend on y, find an *approximate* expression for the average drift of a particle,

$$\lim_{t\to\infty} \frac{\boldsymbol{x}(\boldsymbol{a}, t)}{t}$$

where $\boldsymbol{a}$ is a particle label and A is suitably 'small'. Be precise about what small means.

1.9 *(a)* Suppose that a sealed, insulated container consists of two compartments, and that one of them is filled with an ideal gas and the other is a vacuum. The partition separating the compartments is removed. How does the temperature of the gas change? (Answer: it stays the same. Explain.) Obtain an expression for the final potential temperature, in terms of the initial temperature of the gas and the volumes of the two compartments. Reconcile your answers with the first law of thermodynamics for an ideal gas, that

$$đQ = T\,\mathrm{d}\eta = c_p \frac{\mathrm{d}\theta}{\theta} = \mathrm{d}I + đW = c_v\,\mathrm{d}T + p\,\mathrm{d}\alpha. \tag{P1.6}$$

(b) A dry parcel that is ascending adiabatically through the atmosphere will generally cool as it moves to lower pressure and expands, and its potential temperature stays the same. How can this be consistent with your answer to part (a)?

1.10 Show that adiabatic flow in an ideal gas satisfies $p\rho^{-\gamma}$ = constant, where $\gamma = c_p/c_v$.

1.11 *(a)* Show that for an ideal gas (**??**) is equivalent to (**??**). You may use the Maxwell relation $(\partial\alpha/\partial\eta)_p = (\partial T/\partial p)_\eta$.

(b) Show that for an ideal gas (**??**) is equivalent to (**??**).

1.12 Show that it follows directly from the equation of state, $P = RT/\alpha$, that the internal energy of an ideal gas is a function of temperature only.

Solution: from (**??**) and $p = RT/\alpha$ we have

$$\mathrm{d}\eta = \frac{1}{T}\left(\frac{\partial I}{\partial T}\right)_\alpha \mathrm{d}T + \left[\frac{1}{T}\left(\frac{\partial I}{\partial \alpha}\right)_T + \frac{R}{\alpha}\right]\mathrm{d}\alpha. \tag{P1.7}$$

But, mathematically,

$$\mathrm{d}\eta = \left(\frac{\partial \eta}{\partial T}\right)_\alpha \mathrm{d}T + \left(\frac{\partial \eta}{\partial \alpha}\right)_T \mathrm{d}\alpha. \tag{P1.8}$$

Equating the coefficients of $\mathrm{d}T$ and $\mathrm{d}\alpha$ in these two expressions gives

$$\left(\frac{\partial \eta}{\partial T}\right)_\alpha = \frac{1}{T}\left(\frac{\partial I}{\partial T}\right)_\alpha \qquad \text{and} \qquad \left(\frac{\partial \eta}{\partial \alpha}\right)_T = \frac{1}{T}\left(\frac{\partial I}{\partial \alpha}\right)_T + \frac{R}{\alpha}. \tag{P1.9}$$

Noting that $\partial^2\eta/(\partial\alpha\partial T) = \partial^2\eta/(\partial T\partial\alpha)$ we obtain

$$\frac{\partial^2 I}{\partial\alpha\partial T} = \frac{\partial^2 I}{\partial T\partial\alpha} - \frac{1}{T}\left(\frac{\partial I}{\partial \alpha}\right)_T. \tag{P1.10}$$

Thus, $(\partial I/\partial\alpha)_T = 0$. Because, in general, the internal energy may be considered either a function of temperature and density or temperature and pressure, this proves that for an ideal gas the internal energy is a function *only* of temperature.

1.13 Show that it follows directly from the equation of state $P = RT/\alpha$, that for an ideal gas the heat capacity at constant volume, c_v, is, at most, a function of temperature.

1.14 Show that for an ideal gas

$$T\,\mathrm{d}\eta = c_v \mathrm{d}T + p\mathrm{d}\alpha. \tag{P1.11}$$

and that its internal energy is given by $I = \int c_v \mathrm{d}T$.
Solution: let us regard η as a function of T and α, where α is the specific volume $1/\rho$. Then

$$\begin{aligned} T\,\mathrm{d}\eta &= T\left(\frac{\partial \eta}{\partial T}\right)_\alpha \mathrm{d}T + T\left(\frac{\partial \eta}{\partial \alpha}\right)_T \mathrm{d}\alpha \\ &= c_v\,\mathrm{d}T + T\left(\frac{\partial \eta}{\partial \alpha}\right)_T \mathrm{d}\alpha \end{aligned} \tag{P1.12}$$

by definition of c_v. For an ideal gas the internal energy is a function of temperature alone (problem 1.12), so that using (**??**) the pressure of a fluid $p = T\left(\partial\eta/\partial\alpha\right)_I = T\left(\partial\eta/\partial\alpha\right)_T$ and (P1.12) becomes

$$T\,\mathrm{d}\eta = c_v\,\mathrm{d}T + p\,\mathrm{d}\alpha \tag{P1.13}$$

But, *in general*, the fundamental thermodynamic relation is

$$T\,\mathrm{d}\eta = \mathrm{d}I + p\,\mathrm{d}\alpha. \tag{P1.14}$$

The terms on the right-hand side of (P1.13) are identifiable as the change in the internal energy and the work done on a fluid, and so $dI = c_v\,\mathrm{d}T$. The heat capacity need not necessarily be constant, although for air it very nearly is, but it must be a function of temperature only.

1.15 *(a)* Beginning with the expression for potential temperature for a simple ideal gas, $\theta = T(p_R/p)^\kappa$, where $\kappa = R/c_p$, show that

$$\mathrm{d}\theta = (\theta/T)(\mathrm{d}T - (\alpha/c_p)\,\mathrm{d}p), \tag{P1.15}$$

and that the first law of thermodynamics may be written as

$$đQ = T\,\mathrm{d}\eta = c_p(T/\theta)\mathrm{d}\theta. \tag{P1.16}$$

(b) Verify that $\mathrm{d}\eta = c_p(p_R,\theta)\mathrm{d}\theta/\theta$, for a fluid parcel of constant composition that obeys (**??**).

1.16 Obtain an expression for the Gibbs function of a simple ideal gas, in terms of pressure and temperature.

1.17 From (??) derive the conventional equation of state for an ideal gas, and obtain expressions for the heat capacities.

1.18 A parcel of water is added to the ocean surface that is denser than any water in the ocean. Suppose the parcel sinks adiabatically to the ocean bottom. Estimate the change in temperature that the parcel undergoes, being explicit about the assumptions you make.

1.19 Consider an ocean at rest with known vertical profiles of potential temperature and salinity, $\theta(z)$ and $S(z)$. Suppose that we also know the equation of state in the form $\rho = \rho(\theta, S, p)$. Obtain an expression for the buoyancy frequency. Check your expression by substituting the equation of state for an ideal gas and recovering a known expression for the buoyancy frequency.

1.20 Consider a liquid, sitting in a container, with a free surface at the top (at $z = H$). The liquid obeys the equation of state $\rho = \rho_0[1 - \beta_T(T - T_0)]$, and its internal energy, I, is given by $I = c_p T$. Suppose that the fluid is heated, and its temperature rises uniformly by ΔT, and the free surface rises by a small amount ΔH. Obtain an expression for the ratio of the change in internal energy to the change in gravitational potential energy (GPE) of the ocean, and show that it is related to the scale height (??b). If global warming increases the ocean temperature by 4 K, what is the ratio of the change of GPE to the change of I? Estimate also the average rise in sea level due to thermal expansion.
Partial solution: the change in internal energy and in GPE are

$$\Delta I = c_p \rho_1 H_1 (T_2 - T_1), \qquad \Delta GPE = \rho_1 g H_1 (H_2 - H_1)/2 = \rho_1 g H_1 \beta_T H_1 (T_2 - T_1)/2. \quad \text{(P1.17)}$$

(Derive these. Use mass conservation where necessary. The subscripts 1 and 2 denote initial and final states.) Hence $\Delta GPE/\Delta I = g\beta_T H_1/2c_p$.

1.21 ♦ Obtain an expression, in terms of temperature and pressure, for the potential temperature of a van der Waals gas, with equation of state $(p + a/\alpha^2)(\alpha - b) = RT$, where a and b are constants. Show that it reduces to the expression for an ideal gas in the limit $a \to 0, b \to 0$.

PROBLEM SET 2

GFD and Rotating Fluids

2.1 Show that for an ideal gas in hydrostatic balance, changes in dry static energy ($M = c_p T + gz$) and potential temperature (θ) are related by $\delta M = c_p(T/\theta)\delta\theta$. (The quantity $c_p T/\theta$ is known as the 'Exner function', and is denoted Π.)

2.2 For an ideal gas in hydrostatic balance, show that:

(a) The integral of the potential plus internal energy from the surface to the top of the atmosphere [$\int (P + I)\,\mathrm{d}p$] is is equal to its enthalpy;
(b) $\mathrm{d}\sigma/\mathrm{d}z = c_p(T/\theta)\,\mathrm{d}\theta/\mathrm{d}z$, where $\sigma = I + p\alpha + \Phi$ is the dry static energy;
(c) The following expressions for the pressure gradient force are all equal (even without hydrostatic balance):

$$-\frac{1}{\rho}\nabla p = -\theta\,\nabla\Pi = -\frac{c_s^2}{\rho\theta}\nabla(\rho\theta), \qquad \text{(P2.1)}$$

where $\Pi = c_p T/\theta$ is the Exner function.
(d) Show that item *(a)* also holds for a gas with an arbitrary equation of state.

2.3 Show that, without approximation, the unforced, inviscid momentum equation may be written in the forms

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = T\nabla\eta - \nabla(p\alpha + I) \qquad \text{(P2.2)}$$

and

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = T\nabla\eta - \nabla B \qquad \text{(P2.3)}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$, η is the specific entropy ($\mathrm{d}\eta = c_p\,\mathrm{d}\ln\theta$) and $B = I + \boldsymbol{v}^2/2 + p\alpha$ where I is the internal energy per unit mass.
Hint: First show that $T\nabla\eta = \nabla I + p\nabla\alpha$, and note also the vector identity $\boldsymbol{v}\times(\nabla\times\boldsymbol{v}) = \frac{1}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v}) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$.

2.4 Consider two-dimensional fluid flow in a rotating frame of reference on the f-plane. Linearize the equations about a state of rest.

(a) Ignore the pressure term and determine the general solution to the resulting equations. Show that the speed of fluid parcels is constant. Show that the trajectory of the fluid parcels is a circle with radius $|U|/f$, where $|U|$ is the fluid speed.

(b) What is the period of oscillation of a fluid parcel?

(c) ◆ If parcels travel in straight lines in inertial frames, why is the answer to (b) not the same as the period of rotation of the frame of reference? [To answer this fully you need to understand the dynamics underlying inertial oscillations and inertia circles. See **??** and **?**.]

2.5 A fluid at rest evidently satisfies the hydrostatic relation, which says that the pressure at the surface is given by the weight of the fluid above it. Now consider a *deep* atmosphere on a spherical planet. A unit cross-sectional area at the planet's surface lies beneath a column of fluid whose cross-section increases with height, because the total area of the atmosphere increases with distance away from the centre of the planet. Is the pressure at the surface still given by the hydrostatic relation, or is it greater than this because of the increased mass of fluid in the column? If it is still given by the hydrostatic relation, then the pressure at the surface, integrated over the entire area of the planet, is less than the total weight of the fluid; resolve this paradox. But if the pressure at the surface is greater than that implied by hydrostatic balance, explain how the hydrostatic relation fails.

2.6 In a self-gravitating spherical fluid, like a star, hydrostatic balance may be written

$$\frac{\partial p}{\partial r} = -\frac{GM(r)}{r^2}\rho, \tag{P2.4}$$

where $M(r)$ is the mass interior to a sphere of radius r, and G is a constant. Obtain an expression for the pressure as a function of radius when the fluid (a) has constant density, and (b) is an isothermal ideal gas (if possible). The star is of radius a.

2.7 At what latitude is the angle between the direction of Newtonian gravity (due solely to the mass of the Earth) and that of effective gravity (Newtonian gravity plus centrifugal terms) the largest? At what latitudes, if any, is this angle zero?

2.8 ◆ Write the momentum equations in true spherical coordinates, including the centrifugal and gravitational terms. Show that for reasonable values of the wind, the dominant balance in the meridional component of this equation involves a balance between centrifugal and pressure gradient terms. Can this balance be subtracted out of the equations in a sensible way, so leaving a useful horizontal momentum equation that involves the Coriolis and acceleration terms? If so, obtain a closed set of equations for the flow this way. Discuss the pros and cons of this approach versus the geometric approximation discussed in section **??**.

2.9 For an ideal gas show that the expressions (**??**) and (**??**) are equivalent.

2.10 Consider an ocean at rest with known vertical profiles of potential temperature and salinity, $\theta(z)$ and $S(z)$. Suppose we also know the equation of state in the form $\rho = \rho(\theta, S, p)$. Obtain an expression for the buoyancy frequency. Check your expression by substituting the equation of state for an ideal gas and recovering a known expression for the buoyancy frequency.

2.11 (a) The *geopotential height* is the height of a given pressure level. Show that in an atmosphere with a uniform lapse rate (i.e., $\mathrm{d}T/\mathrm{d}z = \Gamma =$ constant) the geopotential height at a pressure p is given by

$$z = \frac{T_0}{\Gamma}\left[1 - \left(\frac{p_0}{p}\right)^{-R\Gamma/g}\right] \tag{P2.5}$$

where T_0 is the temperature at $z = 0$.

(*b*) In an isothermal atmosphere, obtain an expression for the geopotential height as function of pressure, and show that this is consistent with the expression (P2.5) in the appropriate limit.

2.12 Show that the inviscid, adiabatic, hydrostatic primitive equations for a compressible fluid conserve a form of energy (kinetic plus potential plus internal), and that the kinetic energy has no contribution from the vertical velocity. You may assume Cartesian geometry and a uniform gravitational field in the vertical direction.

2.13 Consider the simple Boussinesq equations, $\mathrm{D}\boldsymbol{v}/\mathrm{D}t = -\nabla\phi + \mathbf{k}b + \nu\nabla^2\boldsymbol{v}$, $\nabla\cdot\boldsymbol{v} = 0$, $\mathrm{D}b/\mathrm{D}t = Q + \kappa\nabla^2 b$. Obtain an energy equation similar to (**??**), but now with the terms on the right-hand side that represent viscous and diabatic effects. Over a closed volume, show that the dissipation of kinetic energy is balanced by a buoyancy source. Also show that, in a statistically steady state, the heating must occur at a lower level than the cooling if a kinetic-energy dissipating circulation is to be maintained.

2.14 ♦ Suppose a fluid is contained in a closed container, with insulating sidewalls, and heated from below and cooled from above. The heating and cooling are adjusted so that there is no net energy flux into the fluid. Let us also suppose that any viscous dissipation of kinetic energy is returned as heating, so the total energy of the fluid is exactly constant. Suppose the fluid starts out at rest and at a uniform temperature, and the heating and cooling are then turned on. A very short time afterwards, the fluid is lighter at the bottom and heavier at the top; that is, its potential energy has increased. Where has this energy come from? Discuss this paradox for both a compressible fluid (e.g., an ideal gas) and for a simple Boussinesq fluid.

2.15 Consider a rapidly rotating (i.e., in near geostrophic balance) Boussinesq fluid on the f-plane.

(*a*) Show that the pressure divided by the density scales as $\phi \sim fUL$.
(*b*) Show that the horizontal divergence of the geostrophic wind vanishes. Thus, argue that the scaling $W \sim UH/L$ is an *overestimate* for the magnitude of the vertical velocity. (Optional extra: obtain a scaling estimate for the magnitude of vertical velocity in rapidly rotating flow.)
(*c*) Using these results, or otherwise, discuss whether hydrostatic balance is more or less likely to hold in a rotating flow than in non-rotating flow.

2.16 Estimate the size of the zonal wind 5 km above the surface in the mid-latitude atmosphere in summer and winter using (approximate) values for the meridional temperature gradient in the atmosphere. Also estimate the shear corresponding to the pole–equator temperature gradient in the ocean.

2.17 Using approximate but realistic values for the observed stratification, what is the buoyancy period for (a) the mid-latitude troposphere, (b) the stratosphere, (c) the oceanic thermocline, (d) the oceanic abyss?

2.18 Consider a dry, hydrostatic, ideal-gas atmosphere whose lapse rate is one of constant potential temperature. What is its vertical extent? That is, at what height does the density vanish? Is this a problem for the anelastic approximation discussed in the text?

2.19 Show that for an ideal gas, the expressions (**??**), (**??**), (**??**) are all equivalent, and express N^2 terms of the temperature lapse rate, $\partial T/\partial z$.

2.20 ♦ Calculate an approximate but reasonably accurate expression for the buoyancy equation for seawater. (From notes by R. de Szoeke)

Solution (i): the buoyancy frequency is given by

$$N^2 = -\frac{g}{\rho}\left(\frac{\partial \rho_{pot}}{\partial z}\right)_{\text{env}} = \frac{g}{\alpha}\left(\frac{\partial \alpha_{pot}}{\partial z}\right)_{\text{env}} = -\frac{g^2}{\alpha^2}\left(\frac{\partial \alpha_{pot}}{\partial p}\right)_{\text{env}} \tag{P2.6}$$

where $\alpha_{pot} = \alpha(\theta, S, p_R)$ is the potential density, and p_R a reference pressure. From (**??**)

$$\alpha_{pot} = \alpha_0\left[1 - \frac{\alpha_0}{c_0^2}p_R + \beta_T(1+\gamma^* p_R)\theta' + \frac{1}{2}\beta_T^*\theta'^2 - \beta_S(S - S_0)\right]. \tag{P2.7}$$

Using this and (P2.6) we obtain the buoyancy frequency,

$$N^2 = -\frac{g^2}{\alpha^2}\alpha_0\left[\beta_T\left(1 + \gamma p_R + \frac{\beta_T^*}{\beta_T}\theta\right)\left(\frac{\partial \theta}{\partial p}\right)_{\text{env}} - \beta_S\left(\frac{\partial S}{\partial p}\right)_{\text{env}}\right], \tag{P2.8}$$

although we must substitute local pressure for the reference pressure p_R. (Why?)

Solution (ii): the sound speed is given by

$$c_s^{-2} = -\frac{1}{\alpha^2}\left(\frac{\partial \alpha}{\partial p}\right)_{\theta,S} = \frac{1}{\alpha^2}\left(\frac{\alpha_0^2}{c_0^2} - \gamma\alpha_1\theta\right) \tag{P2.9}$$

and, using (P2.6) and (**??**) the square of the buoyancy frequency may be written

$$N^2 = \frac{g}{\alpha}\left(\frac{\partial \alpha}{\partial z}\right)_{\text{env}} - \frac{g^2}{c_s^2} = -\frac{g^2}{\alpha^2}\left[\left(\frac{\partial \alpha}{\partial p}\right)_{\text{env}} + \frac{\alpha^2}{c_s^2}\right]. \tag{P2.10}$$

Using (**??**), (P2.9) and (P2.10) we recover (P2.8), although now with p explicitly in place of p_R.

2.21 *(a)* Use the chain rule to show that the horizontal gradients of a field in height coordinates and in ξ coordinates are related by

$$\nabla_z\Psi = \nabla_\xi\Psi - (\partial\Psi/\partial\xi)(\partial\xi/\partial z)\nabla_\xi z. \tag{P2.11}$$

(b) Show that w, the vertical velocity in height coordinates, may be expressed in ξ coordinates as

$$w = \mathrm{D}z/\mathrm{D}t = (\partial z/\partial t)_\xi + \boldsymbol{u}\cdot\nabla_\xi z + \dot{\xi}\partial z/\partial\xi. \tag{P2.12}$$

(c) Use the above expressions to verify (**??**), the expression for the material derivative in ξ coordinates.

2.22 Begin with the mass conservation in height coordinates, namely $\mathrm{D}\rho/\mathrm{D}t + \rho\nabla\cdot\boldsymbol{v} = 0$. Transform this into pressure coordinates using the chain rule (or otherwise) and derive the mass conservation equation in the form $\nabla_p\cdot\boldsymbol{u} + \partial\omega/\partial p = 0$.

2.23 ♦ Starting with the primitive equations in pressure coordinates, derive the form of the primitive equations of motion in sigma-pressure coordinates. In particular, show that the prognostic equation for surface pressure is,

$$\frac{\partial p_s}{\partial t} + \nabla\cdot(p_s\boldsymbol{u}) + p_s\frac{\partial\dot{\sigma}}{\partial\sigma} = 0 \tag{P2.13}$$

and that hydrostatic balance may be written $\partial\Phi/\partial\sigma = -RT/\sigma$.

2.24 Starting with the primitive equations in pressure coordinates, derive the form of the primitive equations of motion in log-pressure coordinates in which $Z = -H \ln(p/p_r)$ is the vertical coordinate. Here, H is a reference height (e.g., a scale height RT_r/g where T_r is a typical or an average temperature) and p_r is a reference pressure (e.g., 1000 mb). In particular, show that if the 'vertical velocity' is $W = DZ/Dt$ then $W = -H\omega/p$ and that

$$\frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p}\left(\frac{pW}{H}\right) = \frac{\partial W}{\partial Z} - \frac{W}{H}. \tag{P2.14}$$

and obtain the mass conservation equation (**??**). Show that this can be written in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_s}\frac{\partial}{\partial Z}(\rho_s W) = 0, \tag{P2.15}$$

where $\rho_s = \rho_r \exp(-Z/H)$.

2.25 *(a)* Prove that the argument of the square root in (**??**) is always positive.
Solution: The largest value or the argument occurs when $m = 0$ and $k^2 = 1/(4H^2)$. The argument is then $1 - 4H^2N^2/c_s^2$. But $c_s^2 = \gamma RT_0 = \gamma gH$ and $N^2 = g\kappa/H$ so that $4N^2H^2/c_s^2 = 4\kappa/\gamma \approx 0.8$.
(b) ◆ This argument seems to depend on the parameters in the ideal gas equation of state. Is it more general than this? Is a natural system possible for which the argument is negative, and if so what physical interpretation could one ascribe to the situation?

2.26 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -A\mathrm{e}^{-(r/\lambda)^2}(y\,\mathbf{i} - x\,\mathbf{j}) \tag{P2.16}$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f/\partial y$ and constant ocean depth H. In the ocean, find (a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

2.27 ◆ In an atmospheric Ekman layer on the f-plane let us write the momentum equation as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_a}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{P2.17}$$

where $\boldsymbol{\tau} = A\rho_a \partial \boldsymbol{u}/\partial z$ and A is a constant eddy viscosity coefficient. An *independent* formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \boldsymbol{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\boldsymbol{u}_g = U\mathbf{i}$.

(a) Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.
(b) Find the vertically integrated horizontal mass flux caused by the boundary layer.
(c) When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this consistent with Newton's third law.
(d) Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a/ρ_o, where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\widehat{u} = u + \mathrm{i}v$ and $\widehat{u}_g = u_g + \mathrm{i}v_g$. The Ekman layer equation, (**??**), may then be written as

$$A\frac{\partial^2 \widehat{U}}{\partial z^2} = \mathrm{i}\, f\widehat{U}, \tag{P2.18}$$

where $\widehat{U} = \widehat{u} - \widehat{u}_g$. The solution to this is

$$\widehat{u} - \widehat{u}_g = [\widehat{u}(0) - \widehat{u}_g] \exp\left[-\frac{(1+\mathrm{i})z}{d}\right], \tag{P2.19}$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial\widehat{u}/\partial z = (C/A)\widehat{u}$; applying this gives $[\widehat{u}(0) - \widehat{u}_g]\exp(\mathrm{i}\pi/4) = -Cd\widehat{u}(0)/(\sqrt{2}A)$, from which we obtain $\widehat{u}(0)$, and the rest of the solution follows. We may also obtain a solution using the same method that was used to obtain (**??**).

2.28 *The logarithmic boundary layer*
Close to ground rotational effects are unimportant and small-scale turbulence generates a *mixed layer.* In this layer, assume that the stress is constant and that it can be parameterized by an eddy diffusivity the size of which is proportional to the distance from the surface. Show that the velocity then varies logarithmically with height.
Solution: Write the stress as $\tau = \rho_0 u^{*2}$ where the constant u^* is called the friction velocity. Using the eddy diffusivity hypothesis this stress is given by

$$\tau = \rho_0 u^{*2} = \rho_0 A\frac{\partial u}{\partial z} \quad \text{where} \quad A = u^* kz, \tag{P2.20}$$

where k is von Karman's ('universal') constant (approximately equal to 0.4). From (P2.20) we have $\partial u/\partial z = u^*/(kz)$ which integrates to give $u = (u^*/k)\ln(z/z_0)$. The parameter z_0 is known as the roughness length.

2.29 Consider two-dimensional compressible flow, for which the mass continuity equation is

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0. \tag{P2.21}$$

and the momentum equation is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -g\nabla h. \tag{P2.22}$$

Obtain the vorticity and divergence equations for this system. Show that the vorticity equation is not closed by itself, and that the divergence equation reduces to gradient wind balance in the limit of small Rossby number if f is constant.

2.30 *(a)* Consider steady inviscid circular flow around a pressure minimum. Draw a diagram (which will have two panels) illustrating the two possible balances between the pressure, Coriolis and centrifugal forces. Repeat this for flow round a pressure maximum.

(b) From (**??**b), show that physically realizable solutions arise only when

$$R\frac{\partial\phi}{\partial n} < \frac{f^2R^2}{4}. \tag{P2.23}$$

In particular, for steady flow around a region of high pressure show that both $\partial\phi/\partial n$ and R are negative, and so that we must have

$$\left|\frac{\partial\phi}{\partial n}\right| < \frac{|R|f^2}{4}. \tag{P2.24}$$

Thus, the deduce that the pressure gradient in the centre of a high pressure region is flat.

(c) Show that for steady flow around a low pressure region, R and $\partial\phi/\partial n$ always take different signs. Thus, infer that there is no similar restriction on the pressure gradient at the center of a low pressure region.

PROBLEM SET 3

Shallow Water and Geostrophic Adjustment

3.1 Derive the appropriate shallow water equations for a single moving layer of fluid of density ρ_1 above a rigid floor, and where above the moving fluid is a stationary fluid of density ρ_0, where $\rho_0 < \rho_1$. Show that as $(\rho_0/\rho_1) \to 0$ the usual shallow water equations emerge.

3.2 *(a)* Model the atmosphere as two immiscible, 'shallow water' fluids of different density stacked one above the other. Using reasonable values for any required physical parameters, estimate the displacement of the interfacial surface associated with a pole–equator temperature gradient of 60 K.

(b) Similarly estimate an interfacial displacement in the ocean associated with a temperature gradient of 20 K over a distance of 4000 km. (This is a crude representation of the main oceanic thermocline.)

3.3 ◆ For a shallow water fluid the energy equation, (**??**), has the form $\partial E/\partial t + \nabla\cdot[\boldsymbol{v}(E+gh^2/2)] = 0$. But for a compressible fluid, the corresponding energy equation, (**??**), has the form $\partial E/\partial t + \nabla\cdot[\boldsymbol{v}(E+p)] = 0$. In a shallow water fluid, $p \neq gh^2/2$ at a point so these equations are superficially different. Explain this and reconcile the two forms. (Hint: What is the average pressure in a fluid column?)

3.4 ◆ Can the shallow water equations for an incompressible fluid be derived by way of an asymptotic expansion in the aspect ratio? If so, do it. That is, without assuming hydrostasy ab initio, expand the Euler equations with a free surface in a small parameter equal to the ratio of the depth of the fluid to the horizontal scale of the motion, and so obtain the shallow water equations.

3.5 The inviscid shallow water equations, whether rotating or not, can support gravity waves of arbitrarily short wavelengths. For sufficiently high wavenumber, the wavelength will be shorter than the depth of the fluid. Is this consistent with an asymptotic nature of the shallow water equations? Discuss.

3.6 Show that the vertical velocity within a shallow water system is given by

$$w = \frac{z-\eta_b}{h}\frac{\mathrm{D}h}{\mathrm{D}t} + \frac{\mathrm{D}\eta_b}{\mathrm{D}t}. \tag{P3.1}$$

Interpret this result, showing that it gives sensible answers at the top and bottom of the fluid layer.

3.7 What is the appropriate generalization of (**??**) to two dimensions? Suppose that at time $t = 0$ the height field is given by a Gaussian distribution $h' = A\mathrm{e}^{-r^2/\sigma^2}$, where $r^2 = x^2 + y^2$. What is the subsequent evolution of this, in the linear approximation? [Hint: look in the book *Methods of Theoretical Physics* by Morse and Feshbach.]

3.8 In an adiabatic shallow water fluid in a rotating reference frame show that the potential vorticity conservation law is

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{\eta - h_b} = 0 \tag{P3.2}$$

where η is the height of the free surface and h_b is the height of the bottom topography, both referenced to the same flat surface.

(a) A cylindrical column of air at 30° latitude with radius 100 km expands horizontally to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after the expansion?

(b) An air column at 60° N with zero relative vorticity ($\zeta = 0$) stretches from the surface to the tropopause, which we assume is a rigid lid, at 10 km. The air column moves zonally on to a plateau 2.5 km high. What is its relative vorticity? Suppose it then moves southwards to 30° N. What is its vorticity? (Assume that the density is constant.)

3.9 ◆ In the long-wave limit of Poincaré waves, fluid parcels behave as free-agents; that is, like free solid particles moving in a rotating frame unencumbered by pressure forces. Why then, is their frequency given by $\omega = f = 2\Omega$ where Ω is the rotation rate of the coordinate system, and not by Ω itself? Do particles that are stationary or move in a straight line in the inertial frame of reference satisfy the dispersion relationship for Poincaré waves in this limit? Explain. [See also **??** and **?**.]

3.10 Linearize the f-plane shallow water system about a state of rest. Suppose that there is an initial disturbance given in the general form

$$\eta = \iint \widetilde{\eta}_{k,l}\,\mathrm{e}^{\mathrm{i}(kx+ly)}\,\mathrm{d}k\,\mathrm{d}l, \tag{P3.3}$$

where η is the deviation surface height and the Fourier coefficients $\widetilde{\eta}_{k,l}$ are given, and that the initial velocity is zero.

(a) Obtain the geopotential field at the completion of geostrophic adjustment, and show that the deformation scale is a natural length scale in the problem.

(b) Show that the change in total energy during the adjustment is always less than or equal to zero. Neglect any initial divergence.
N.B. Because the problem is linear, the Fourier modes do not interact.

3.11 One way of assimilating observations into numerical models is via a method known as 'nudging' or 'robust diagnostics'. In this method, a relaxation term is added to the right-hand side of an evolution equation, relaxing the field back to observations. Thus, for example, $\mathrm{D}\phi/\mathrm{D}t$ = other terms + $\lambda(\phi_{\mathrm{obs}} - \phi)$. If ϕ is potential temperature, is this method likely to work better in the ocean or in the atmosphere, as regards reproducing the large scale mean potential temperature and velocity fields? What steps might be taken to improve the method?

3.12 *Geostrophic adjustment of a velocity jump*
Consider the evolution of the linearized f-plane shallow water equations in an infinite domain. Suppose that initially the fluid surface is flat, the zonal velocity is zero and the meridional velocity is given by

$$v(x) = v_0\,\mathrm{sgn}(x) \tag{P3.4}$$

(a) Find the equilibrium height and velocity fields at $t = \infty$.
(b) What are the initial and final kinetic and potential energies?

Partial Solution:
The potential vorticity is $q = \zeta - f_0\eta/H$, so that the initial and final state are

$$q = 2v_0\,\delta(x). \tag{P3.5}$$

(Why?) The final state streamfunction is thus given by $\left(\partial^2/\partial x^2 - L_d^{-2}\right)\psi = q$, with solution $\psi = \psi_0 \exp(x/L_d)$ and $\psi = \psi_0 \exp(-x/L_d)$ for $x < 0$ and $x > 0$, where $\psi_0 = -L_d v_0$ (why?), and $\eta = f_0\psi/g$. The energy is $E = \int (Hv^2 + g\eta^2)/2\,\mathrm{d}\boldsymbol{x}$. The initial KE is infinite, the initial PE is zero, and the final state has $PE = KE = gL_d\eta_0^2/4$; that is, the energy is equipartitioned between kinetic and potential energies.

3.13 In the shallow water equations show that, if the flow is approximately geostrophically balanced, the energy at large scales is predominantly potential energy and that energy at small scales is predominantly kinetic energy. Define precisely what 'large scale' and 'small scale' mean in this context.

3.14 In the shallow water geostrophic adjustment problem, show that at large scales the velocity adjusts to the height field, and that at small scales the height field adjusts to the velocity field.

3.15 ♦ Consider the problem of minimizing the full energy [i.e., $\int (h\boldsymbol{u}^2) + g\eta^2)\,\mathrm{d}\boldsymbol{x}$], given the potential vorticity field $q(x, y) = (\zeta + f)/h$. Show that the balance relations analogous to (**??**) are $uh = -\partial(Bq^{-1})/\partial y$ and $vh = \partial(Bq^{-1})/\partial x$ where B is the Bernoulli function $B = g\eta + \boldsymbol{u}^2/2$. Show that steady flow does not necessarily satisfy these equations. Discuss.

3.16 Using realistic values for temperature, velocity, etc., calculate *approximate* values for the total potential energy, the available potential energy and the kinetic energy, of either a hemisphere in the atmosphere or an ocean basin.